Fundamental Notions in Algebra – Exercise No. 7

1. Let A and B be algebras over k, and let $D \subset A$ and $E \subset B$ k-subalgebras. Show that $C_{A\otimes_k B}(D \otimes_k E) = C_A(D) \otimes_k C_B(E)$.

2. Let S be an algebra over k of dimension n^2. Show that the following are equivalent:

   (a) S is a simple central algebra over k;

   (b) $S_{\bar{k}} \cong \mathrm{Mat}_n(\bar{k})$, where $\bar{k}$ is the algebraic closure of k;

   (c) $S_L \cong \mathrm{Mat}_n(L)$ for some finite extension L/k with $n \mid [L : k]$.

3. (a) Let D be a division ring, and let F be the minimal division subalgebra of D containing 1. Show that F is a subfield of $Z(D)$, isomorphic either to the field of rational numbers $\mathbb{Q}$ or to the finite field $\mathbb{F}_p$ for some prime p.

   (b) Let D be a division ring, $G \subset D^\times$ be a finite subgroup, and F as in (a). Assume either that G is commutative or that $F = \mathbb{F}_p$ for some p. Show that G is cyclic.

   Hint. Show first that in both cases the subalgebra $E := \left\{ \sum_{g\in G} a_g g \; : \; a_g \in F \right\}$ is a field.

   (c) Give an example of a division algebra D and a non-cyclic finite subgroup $G \subset D^\times$.

4. Show that the conclusion of the Skolem Noether theorem remains true if we assume that S is a simple central artinian algebra over k, and R is simple finite-dimensional algebra over k.

   Hint. Mimic the original proof: show first that the algebra $R \otimes_k S$ is simple and artinian. Next write S in the form $\mathrm{End}_D(V)$. Endow V with two structures of $R \otimes_k S$-modules, one using $f : R \to S$ and the other using $g : R \to S$. Then show that there exists a morphism $j : V \to V$ such that

   - $j \circ d = d \circ j$ for every $d \in D$,
   - $j \circ f(r) = g(r) \circ j$ for every $r \in R$
   - j is either one-to-one or is onto.

   Prove that such j has to be an isomorphism.

5. Let k be a field of characteristic different from 2, and let D be a non-commutative division algebra over k of dimension 4. Show that $Z(D) = k$ and there exist elements $i, j \in D$ such that $\{1, i, j, ij\}$ is a basis of D over k, $ji = -ij$ and $i^2 = a, j^2 = b$ for some $a, b \in k$.

6. (a) Let D be a central finite-dimensional division algebra over k. Show that k-algebra $\mathrm{Mat}_{n_1}(D)$ can be embedded into the k-algebra $\mathrm{Mat}_{n_2}(k)$ if and only if the product $n_1 dim_k(D)$ divides n_2.

(b) Let D_1, D_2, D_3 be central finite-dimensional division algebras over k such that $D_1 \otimes_k D_2^{op} \cong \mathrm{Mat}_{n_3}(D_3)$. Show that k-algebra $\mathrm{Mat}_{n_1}(D_1)$ can be embedded into the k-algebra $\mathrm{Mat}_{n_2}(D_2)$ if and only if the product $n_1 dim_k(D)$ divides $n_2 n_3$.

Hint. Write $Mat_{n_2}(D_2)$ in the form $\mathrm{End}_{D_2^{op}}(V)$ and try to embed $\mathrm{Mat}_{n_1}(D_1) \otimes_k D_2^{op}$ into $\mathrm{End}_k(V)$.