THE SPECIAL FIBER OF A PARAHORIC SUBGROUP

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ABSTRACT. These are notes for the fifth lecture in the Bruhat–Tits masters' student seminar at the University of Bonn in Fall 2022. In previous lectures, we constructed the integral models of parahoric subgroup. In this lecture we reap the fruits of this construction, studying the special fibers of these integral models. The maximal reductive quotients of such are finite reductive groups that play an important role in representation theory. After reviewing the relative identity component, a notion that has been used in the seminar but not adequately explained, we show how to compute the root data of these finite reductive groups when the ambient group is quasi-split. This calculation is a beautiful application of affine root systems. In rank two one can draw nice pictures, some of which we reproduce in these notes.

NOTATION

Let K be a complete discretely valued field with algebraically closed residue field $\mathfrak{f}$. The assumption on $\mathfrak{f}$ is not essential; it accommodates our failure to discuss unramified descent. The next two talks will rectify this failure.

For brevity, we write "the book" to refer to the May 2022 draft of Kaletha and Prasad's book on Bruhat–Tits theory, the standard reference for our seminar.

1. RELATIVE IDENTITY COMPONENT

1.1. **Generalities.** Recall that the identity component G^0 of a K-group scheme G locally of finite type (for K is an arbitrary field) is the connected component of the underlying topological space of G that contains the identity. Since G^0 is an open subgroup of G, it inherits a scheme structure. If in addition G is quasi-compact, in other words, of finite type, then G^0 has finite index in G.

In the theory of integral models of group schemes, one could define the identity component of a group scheme locally of finite type in the same way, as the connected component of the identity in the underlying topological space. It turns out to be more useful to work with a relative version of this subgroup, where one requires all fibers to be connected, not just the generic fiber.

Example 1. Let π be a uniformizer of K, suppose $p \neq 2$, and let $\mathscr{G}$ be the norm-one torus defined by the equation $x^2 + \pi y^2 = 1$, which we saw earlier in the seminar. Then $\mathscr{G}$ and $\mathscr{G}_K$ are connected but $\mathscr{G}_{\mathfrak{f}} \simeq \mu_2 \times \mathbb{G}_{\mathrm{a}}$ is disconnected.

Definition 2. Let $\mathscr{G}$ be an $\mathcal{O}$-group scheme. The relative identity component of $\mathscr{G}$ is the $\mathcal{O}$-group $\mathscr{G}^0 \colon \mathrm{Sch}_{/\mathcal{O}}^{\mathrm{op}} \to \mathrm{Set}$ defined by

$$\mathscr{G}^0(S) = \{u \in \mathscr{G}(S) : u(S_K) \subseteq \mathscr{G}_K^0 \quad \text{and} \quad u(S_{\mathfrak{f}}) \subseteq \mathscr{G}_{\mathfrak{f}}^0\}.$$

Proposition 3. *Let $\mathscr{G}$ be a $\mathcal{O}$-group scheme.*

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(1) If $\mathscr{G}$ is smooth then $\mathscr{G}^0_K \sqcup \mathscr{G}^0_{\mathfrak{f}}$ is an open subscheme of $\mathscr{G}$ representing $\mathscr{G}^0$.
(2) If, in addition, $\mathscr{G}$ is affine then $\mathscr{G}^0$ is affine.

Proof. The first part is proved in SGA 3 [DG70, VI_B.3.10]. Once one knows that this subset is open, it is relatively easy to see that it represents $\mathscr{G}^0$. The proof of openness is not so obvious, though it is clear when $\mathscr{G}_K$ is connected and $\mathscr{G}_{\mathfrak{f}}$ is affine: then the complement of $\mathscr{G}^0_{\mathfrak{f}}$ in $\mathscr{G}^0_{\mathfrak{f}}$ is closed in $\mathscr{G}_{\mathfrak{f}}$, hence in $\mathscr{G}$.

The second part follows from a more general result of Raynaud, which states that a flat $\mathcal{O}$-group scheme of finite type with affine generic fiber is affine if and only if it is separated [PY06, 3.1]. Since $\mathscr{G}^0$ is an open subscheme of a separated scheme by the first part, $\mathscr{G}^0$ is itself separated, hence affine by Raynaud's criterion. □

1.2. **Application to parahoric groups.** We begin with an illuminating example that naturally arises in Bruhat-Tits theory.

Example 4. Consider the lft-Néron model $\mathbb{G}^{\mathrm{lft}}_{\mathrm{m},K}$ of $\mathbb{G}_{\mathrm{m},K}$. From the universal mapping property, it is easy to see that

$$\mathbb{G}^{\mathrm{lft}}_{\mathrm{m}}(\mathcal{O}) = K^\times.$$

This is much larger than we would expect for an $\mathcal{O}$-scheme, since $\mathbb{G}_{\mathrm{m},\mathcal{O}}(\mathcal{O}) = \mathcal{O}^\times$. In fact, $\mathbb{G}^{\mathrm{lft}}_{\mathrm{m},K}(\mathcal{O})$ admits the following description: it is obtained from the disjoint union of countably many copies of $\mathbb{G}_{\mathrm{m},\mathcal{O}}$ by gluing them together along their generic fibers. In other words, $\mathbb{G}_{\mathrm{m},\mathcal{O}}$ is the pushout of the diagram

$$\mathbb{G}_{\mathrm{m},K} \to \coprod_{i\in\mathbb{Z}} \mathbb{G}_{\mathrm{m},\mathcal{O}}$$

given by inclusion of generic fibers in each integral torus. This is a rather strange $\mathcal{O}$-group scheme: its generic fiber is $\mathbb{G}_{\mathrm{m},K}$ but its special fiber is $\mathbb{Z}\times\mathbb{G}_{\mathrm{m},\mathfrak{f}}$. In particular, $\mathbb{G}^{\mathrm{lft}}_{\mathrm{m}}$ is not of finite type. However, the relative identity component is what one would expect:

$$\left(\mathbb{G}^{\mathrm{lft}}_{\mathrm{m},K}\right)^0 = \mathbb{G}_{\mathrm{m},\mathcal{O}}.$$

Our main application of the relative identity component is to the construction of parahoric integral models. Let $x \in \mathcal{B}(G)$. In the previous lecture, we constructed a smooth integral model $\mathscr{G}^1_x$ of G such that

$$\mathscr{G}^1_x(\mathcal{O}) = G(\mathcal{O})^1_x.$$

Definition 5. The parahoric integral model of G at x is the $\mathcal{O}$-group scheme $\mathscr{G}^0_x$.

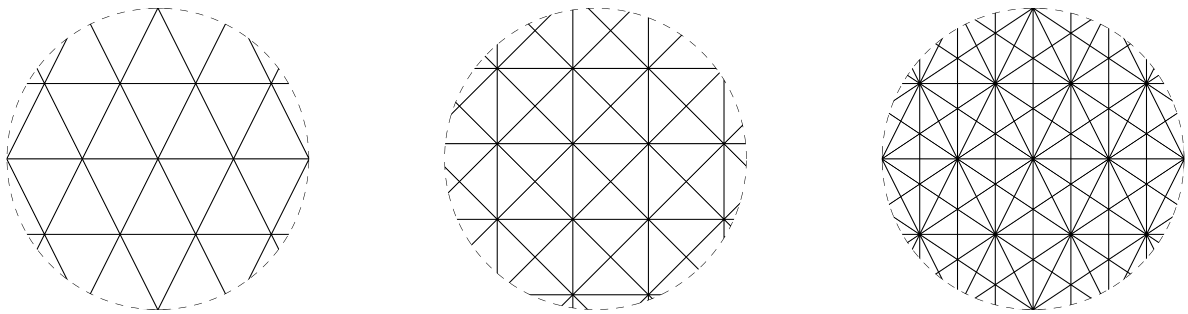
Fact 6. *Let $\mathcal{F}$ be a facet of $\mathcal{B}(G)$ and let $x, y \in \mathcal{F}$. Then $\mathscr{G}^0_x \simeq \mathscr{G}^0_y$ as integral models of G.*

However, the stabilizers of x and y in $G(K)$ are not always the same when x and y lie in the same facet: the stabilizer can grow when x is the barycenter, for instance. We write $\mathscr{G}^0_{\mathcal{F}}$ for the parahoric integral model of Fact 6.

2. The affine root system of a quasi-split group

2.1. **Affine root systems.** In this subsection we recall some facts about affine root systems that will later inform our understanding of special fibers of parahorics.

Let V be a finite-dimensional real vector space and let A be an affine space under V (in other words, a V-torsor). Recall that A^* denotes the vector space of affine functionals on A. Let $\Psi \subseteq A^*$ be an affine root system in A with spherical root system Φ. The vanishing hyperplanes of Φ partition A into facets. The facets of maximal dimension are



Type A_2 Type B_2 Type G_2

FIGURE 1. Split rank-two affine root systems

called chambers. We write $\mathcal{F} \leq \mathcal{F}'$ if $\mathcal{F}$ is contained in the closure of $\mathcal{F}'$. See Figure 1 for a drawing of the hyperplane systems of split rank-two affine root systems.

For $x \in A$, let $\Psi_x := \{\alpha \in \Psi : \alpha(x) = 0\}$. The set Ψ_x is a root system in the vector space obtained from A by basing 0 at x. Moreover, Ψ_x depends only on the facet $\mathcal{F}$ containing x, so we may define $\Psi_\mathcal{F} := \Psi_x$. Alternatively, $\Psi_\mathcal{F} = \{\alpha \in \Psi : \alpha|_\mathcal{F} = 0\}$. The restriction of ∇ to $\Psi_\mathcal{F}$ is injective; let $\Phi_\mathcal{F} \subseteq \Phi$ denote its image, a root system.

Warning 7. If Φ is nonreduced then $\Phi_\mathcal{F}$ need not be a closed subsystem of Φ. This stems ultimately from the fact that B_n is not a closed subsystem of BC_n.

Let $\mathcal{C}$ be a chamber and let $\Psi^0(\mathcal{C})$ be the set of indivisible affine roots that are positive on $\mathcal{C}$ and whose vanishing hyperplanes bound $\mathcal{C}$. The set $\Psi^0(\mathcal{C})$ is a basis of Φ.

Remark 8. There is a general definition of a basis of an affine root system that we omit. The set $\Psi^0(\mathcal{C})$ satisfies the definition. In fact, $\mathcal{C} \mapsto \Psi^0(\mathcal{C})$ is a bijection between chambers of A and bases of Ψ.

Proposition 9. *Let $\mathcal{F}$ be a facet and let $\mathcal{C}$ be a chamber with $\mathcal{F} \leq \mathcal{C}$.*

(1) Then $\Psi^0(\mathcal{C}) \cap \Psi_\mathcal{F}$ is a basis of $\Psi_\mathcal{F}$.

(2) The assignment $\mathcal{C} \mapsto \Psi^0(\mathcal{C}) \cap \Psi_\mathcal{F}$ is a bijection

$$\{\text{chambers } \mathcal{C} : \mathcal{F} \leq \mathcal{C}\} \simeq \{\text{bases of } \Psi_\mathcal{F}\}.$$

(3) $\mathrm{Dyn}(\Psi_\mathcal{F})$ is the subdiagram of $\mathrm{Dyn}(\Psi)$ spanned by the nodes in $\Psi^0_\mathcal{F}(\mathcal{C})$.

Finally, let's describe the parabolic subsets of $\Psi_\mathcal{F}$. Recall that a subset P of a root system Φ is parabolic if it is closed in Φ and $\Phi = P \cup (-P)$. Given facets $\mathcal{F} \leq \mathcal{F}'$, let

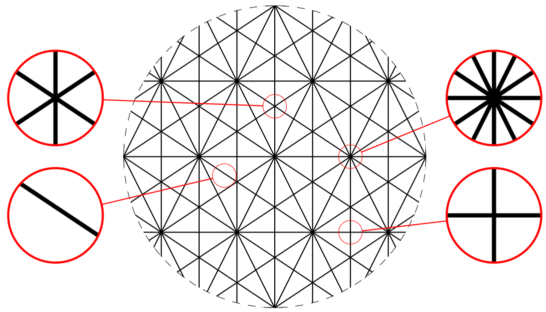
$$\Psi_\mathcal{F}(\mathcal{F}')^+ := \{\alpha \in \Psi_\mathcal{F} : \alpha(\mathcal{F}') \geq 0\}.$$

Proposition 10. *Let $\mathcal{F}$ be a facet. The assignment $\mathcal{F}' \mapsto \Psi_\mathcal{F}(\mathcal{F}')^+$ is a bijection*

$$\{\text{facets } \mathcal{F}' \subseteq A : \mathcal{F} \leq \mathcal{F}'\} \simeq \{\text{parabolic subsets of } \Psi_\mathcal{F}\}.$$

2.2. **Relative root systems.** Since $\dim(K) \leq 1$, the group G is quasi-split. Let S denote a split maximal torus of G and let T be the centralizer of S in G. Although many of the constructions of this lecture are independent of this choice of S, it helps us to "coordinatize" the problem using the root system and root groups.

From S one can construct the relative root system $\Phi = \Phi(G, S) \subseteq X^*(S)$, and from T the absolute root system $\widetilde{\Phi} = \Phi(G, T) \subseteq X^*(T)$. The absolute Galois group acts on these



Type G_2

FIGURE 2. Local hyperplane arrangements for G_2

objects, trivially in the case of S, and the restriction map $X^*(T) \to X^*(S)$ induces a Galois-equivariant surjection $\pi\colon \widetilde{\Phi} \to \Phi$. More simply, relative roots of G may be identified with Galois orbits of absolute roots of G.

Remark 11. The theory of folding gives a recipe for the Dynkin diagram of Φ in terms of the Dynkin diagram of $\widetilde{\Phi}$: take the quotient by the Galois action. The recipe is more complicated, however, when the Galois action stabilizes an edge. One can see by a quick glance at the table of Dynkin diagrams that among the irreducible root systems, the only one where this complication occurs is in type A_{2n}. Not coincidentally, it is the only irreducible absolute root system whose relative root system is nonreduced.

On the group level, the only simply-connected absolutely simple group of this type are odd special unitary groups for a ramified quadratic extension. Although one must add many words to handle the nonreduced case, this extra layer of complexity can be ignored much of the time.

Since G is quasi-split, the Galois action on $X^*(T)$ preserves a set of positive roots. Such an automorphism is said to be pinned. A pinned automorphism induces an automorphism of the (absolute) Dynkin diagram, and this assignment is a bijection between pinned automorphisms (up to Weyl conjugacy) and Dynkin-diagram automorphisms. The group G is completely classified by its root datum together with this pinned Galois action. In fact, there is an equivalence of categories between pinned root data equipped with a Galois action and pinned quasi-split k-groups. The takeaway is that one can combinatorially enumerate and understand quasi-split groups through the Galois action on the roots.

Remark 12. In contrast, over an arbitrary field one cannot hope to classify non-quasi-split reductive groups combinatorially. The inner twists of GL_n, for instance, are in bijection with degree-n central simple algebras over the base field, and such objects are classified by Galois cohomology, which is very sensitive to the arithmetic of the base field.

However, we will see later that in the setting of Bruhat-Tits theory, it is possible to classify even the non-quasi-split groups using combinatorics.

We also have the root groups $(U_a)_{a\in\Phi}$. Recall that a root a is divisible if $a/2 \in \Phi$ and is multipliable if $2a \in \Phi$. These concepts are relevant only when Φ is nonreduced.

Root groups come in two kinds.[1] Let $\tilde{a} \in \pi^{-1}(a)$ and let $K_{\tilde{a}}$ be a field extension of K corresponding to $\tilde{a}$. In other words, $K_{\tilde{a}}$ is the fixed field in some separable closure of K for the stabilizer of $\tilde{a}$ in the absolute Galois group.

First, suppose a is not multipliable ("cases R1 and R3"). Then

$$U_a \simeq \operatorname{Res}_{K_{\tilde{a}}/K} U_{\tilde{a}} \simeq \operatorname{Res}_{K_{\tilde{a}}/K} \mathbb{G}_{\mathrm{a},K_{\tilde{a}}} \simeq \mathbb{G}_{\mathrm{a}}^{[K_{\tilde{a}}:K]}.$$

In particular, $\dim U_a = [K_{\tilde{a}} : K] = |\pi^{-1}(a)|$. Second, suppose that a is multipliable ("case R2") so that $2a$ is divisible. In this case, one has a certain separable quadratic extension $K_{\tilde{b}}/K_{\tilde{a}}$ and a three-dimensional unipotent $K_{\tilde{a}}$-group scheme $U_{K_{\tilde{b}}/K_{\tilde{b}}}$ such that

$$U_a \simeq \operatorname{Res}_{K_{\tilde{a}}/K} U_{K_{\tilde{b}}/K_{\tilde{a}}}.$$

In particular, $\dim U_a = 3[K_{\tilde{a}}/K] = 3|\pi^{-1}(a)|$.

This unipotent group in the second case is defined as follows. To simply the notation, we replace the quadratic extension $K_{\tilde{b}}/K_{\tilde{a}}$ by a quadratic L/K. Let $u \mapsto \bar{u}$ denote the nontrivial K-automorphism of L. The functor of points for $U_{L/K}$ is

$$U_{L/K}(R) = \{u, v \in R \otimes_k \ell : v + \bar{v} = u\bar{u}\}$$

and the group law is $(u, v) \cdot (u', v') = (u + u', v + v' + u\bar{u})$. The central subgroup $\operatorname{Res}^0_{L/K} \mathbb{G}_{\mathrm{a}}$ where $u = 0$ is the group of trace-zero elements in this quadratic extension, a group non-canonically isomorphic to $\mathbb{G}_{\mathrm{a}}$. All in all, we can describe $U_{L/K}$ as a central extension

$$1 \to \operatorname{Res}^0_{L/K} \mathbb{G}_{\mathrm{a}} \to U_{L/K} \to \operatorname{Res}_{L/K} \mathbb{G}_{\mathrm{a}} \to 1.$$

The definition of $U_{L/K}$ is a bit mysterious the way we presented it, but one can work out the definition from first principles by analyzing the root groups of SU_3.

Remark 13. When $p = 2$, there is a very subtle issue with the indexing of the filtration on the group $U_{L/K}$, described in Section 3.2(e) of the book, and which propagates to the definition of the valuation of the relative root datum (Section 6.1). The definition of $U_{L/K}$ gives an indication as to the problem. If K has characteristic zero (for simplicity) and $L = K(\sqrt{a})$ then $\operatorname{tr}_{L/K}(x + y\sqrt{a}) = 2y\sqrt{a}$. When in addition $\operatorname{val}(2) > 0$, the imaginary elements of L are smaller than expected.

Exercise 14. Let G be a split K-group and L/K a finite separable extension. Describe the relative root system of $\operatorname{Res}_{L/K} G_L$.

2.3. **Root systems in the apartment.** Let $\mathcal{A} = \mathcal{A}(G, S)$ be the apartment of $\mathcal{B}(G)$ corresponding to S. In brief, one defines $\mathcal{A}$ as a certain collection of "valuations of a root datum", certain filtrations of the K-points of the root groups whose precise definition we will elide. Of particular importance are the Chevalley valuations, which arise from a choice of Chevalley–Steinberg pinning of G with respect to S. The Euclidean cocharacter lattice $V := X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ acts on the set of valuations for its root groups, and one defines $\mathcal{A}$ as the unique equivalence class under this action that contains a Chevalley valuation.

In particular, the apartment $\mathcal{A}$ is an affine space – in other words, a torsor – for the vector space V. This affine space is an important ingredient in the definition of the affine root system $\Psi = \Psi(G, S)$. Specifically, for any $\psi \in \mathcal{A}^*$ with $\nabla\psi = a \in \Phi$, one uses the valuation to define

$$U_a(K)_\psi := U_a(K)_{x,\psi(x)}$$

[1]Arguably case R3 is slightly different from case R1, though we elide the difference in these notes.

for any $x \in \mathcal{A}$; the definition is independent of the choice of x. Then

$$\Psi := \{\psi \in \mathcal{A}^* \colon U(K)_\psi \subsetneq U(K)_{\psi+} \cdot U_{2\nabla\psi}(K)\}.$$

Here we define $U_{2\nabla\psi} = 1$ whenever $2\nabla\psi \notin \Phi$. The factor $U_{2\nabla\psi}(K)$ in the definition of Ψ is irrelevant when Ψ is reduced.

Example 15. Suppose G is split and the valuation on K is normalized to have value group $\mathbb{Z}$. Then there is an identification $\Psi = \{a + n : a \in \Phi, n \in \mathbb{Z}\}$. Identifying A with V, we can describe the root-group filtration as $U_a(K)_{0,r} = \{c \in K : \mathrm{val}(c) \geq r\}$.

3. Special fibers of parahoric groups

3.1. **Structure.** Given a facet $\mathcal{F}$, let

$$\mathsf{G}_{\mathcal{F}} := \overline{\mathscr{G}}_{\mathcal{F}} / \mathcal{R}_u(\overline{\mathscr{G}}_{\mathcal{F}})$$

denote the maximal reductive quotient of the special fiber of the parahoric $\mathscr{G}_{\mathcal{F}}$. By definition, the group $\mathsf{G}_{\mathcal{F}}$ is reductive, and since we assumed $\mathfrak{f}$ to be algebraically closed, this group is in fact split. Hence $\mathsf{G}_{\mathcal{F}}$ is completely determined by its root datum, which we now describe.

Exercise 16. Typically $\mathsf{G}_{\mathcal{F}}$ is much smaller than the full special fiber $\overline{\mathscr{G}}_{\mathcal{F}}$. Indeed, when $\mathcal{F}$ is a chamber, $\dim \mathsf{G}_{\mathcal{F}} = \dim S$ but $\dim \overline{\mathscr{G}}_{\mathcal{F}} = \dim G$. When $G = \mathrm{SL}_2$ and $\mathcal{F}$ is a chamber, show explicitly that the special fiber of the corresponding Iwahori subgroup is

$$\overline{\mathscr{G}}_{\mathcal{F}} \simeq \mathbb{G}_{\mathrm{a}}^2 \rtimes \mathbb{G}_{\mathrm{m}}.$$

What is the action of $\mathbb{G}_{\mathrm{m}}$? (Hint: describe the Iwahori subgroup by its functor of points.)

Since S is split its special fiber $\overline{S}$ is already reductive: $\overline{S} = \mathsf{S}$. Moreover, the inclusion $S \hookrightarrow G$ extends to a homomorphism $\mathscr{S} \hookrightarrow \mathscr{G}_{\mathcal{F}}$ of integral models. Passage to special fibers induces a homomorphism $\mathsf{S} \to \mathsf{G}_{\mathcal{F}}$. In other words, S is a split torus of $\mathsf{G}_{\mathcal{F}}$. In fact, S is a maximal torus, as one can show using the smoothness of the moduli space of tori in the special fiber. Since $X^*(S)$ is canonically isomorphic to $X^*(\mathsf{S})$, we can naturally interpret $\Phi_{\mathcal{F}}$ as a subset of $X^*(\mathsf{S})$.

Theorem 17. *(1) The root datum of $\mathsf{G}_{\mathcal{F}}$ with respect to S is $(X^*(S), \Phi_{\mathcal{F}}, X_*(S), \Phi_{\mathcal{F}}^\vee)$.*
(2) The root group for $a \in \Phi_{\mathcal{F}}$ is the quotient $\overline{\mathscr{U}}_{a,x,0} / \overline{\mathscr{U}}_{a,x,0+}$.

We can deduce the following corollary from a dimension count.

Corollary 18. *$\mathscr{G}_{\mathcal{F}}^0$ is reductive if and only if $\mathcal{F}$ is a (hyper)special vertex.*[2]

3.2. **Comparing special fibers.** Let $\mathcal{F}$ and $\mathcal{F}'$ be facets. When $\mathcal{F} \leq \mathcal{F}'$, we have an obvious inclusion $G(K)_{\mathcal{F}} \subseteq G(K)_{\mathcal{F}'}$. By properties of smooth integral models that we reviewed in the first lecture, the inclusion gives rise to a map $\mathscr{G}_{\mathcal{F}} \to \mathscr{G}_{\mathcal{F}'}$ of $\mathcal{O}$-groups, and thus a map $\mathscr{G}_{\mathcal{F}}^0 \to \mathscr{G}_{\mathcal{F}'}^0$ of parahoric groups by functoriality of the relative identity component. Passing to special fibers yields a map

$$\bar{\rho}_{\mathcal{F},\mathcal{F}'} \colon \overline{\mathscr{G}}_{\mathcal{F}}^0 \to \overline{\mathscr{G}}_{\mathcal{F}'}^0$$

Example 19. When $\mathcal{F} = \mathcal{C}$ is a chamber, the map $\mathsf{G}_{\mathcal{F}} \to \mathsf{G}_{\mathcal{F}'}$ is the inclusion of the split maximal torus $\mathsf{S} \to \mathsf{G}_{\mathcal{F}'}$ that we used to describe the root datum of $\mathsf{G}_{\mathcal{F}'}$.

[2] A vertex is hyperspecial if it remans special after every unramified extension. But since we assumed $k = K$, every special vertex is automatically hyperspecial.

Warning 20. It is tempting to proceed one step further and say that $\rho_{\mathcal{F},\mathcal{F}'}$ induces a map $\mathsf{G}_{\mathcal{F}} \to \mathsf{G}_{\mathcal{F}'}$ on maximal reductive quotients. However, this deduction is incorrect: formation of the maximal reductive quotient is not functorial with respect to all homomorphisms, only homomorphisms with normal image. The map $\rho_{\mathcal{F},\mathcal{F}'}$ does not usually have this property. Indeed, as we will see shortly, the image of $\rho_{\mathcal{F},\mathcal{F}'}$ is a parabolic subgroup.

Let $\mathfrak{p}_{\mathcal{F}}(\mathcal{F}')$ denote the image of $\rho_{\mathcal{F},\mathcal{F}'}$ in $\mathsf{G}_{\mathcal{F}}$.[3]

Theorem 21. *(1) $\mathfrak{p}_{\mathcal{F}}(\mathcal{F}')$ is a parabolic subgroup of $\mathsf{G}_{\mathcal{F}}$ containing S.*
(2) $\Phi(\mathfrak{p}_{\mathcal{F}}(\mathcal{F}'), \mathsf{S}) = \nabla\big(\Psi_{\mathcal{F}}(\mathcal{F}')\big) \subseteq \Phi_{\mathcal{F}}$.
(3) The assignment $\mathcal{F}' \mapsto \mathfrak{p}_{\mathcal{F}}(\mathcal{F}')$ is a bijection

$$\{\text{facets } \mathcal{F}' \subseteq \mathscr{B}(G) : \mathcal{F} \leq \mathcal{F}'\} \simeq \{\text{parabolic subgroups of } \mathsf{G}_{\mathcal{F}}\}.$$

Although we have fixed a maximal split torus S throughout the discussion, in the last part of the theorem we can finally remove it.

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[PY06] Gopal Prasad and Jiu-Kang Yu, *On quasi-reductive group schemes*, J. Algebraic Geom. **15** (2006), no. 3, 507–549, With an appendix by Brian Conrad. MR 2219847

[3]The book uses this notation for the image in $\overline{\mathscr{G}}^0_{\mathcal{F}}$.