Universal Symmetries: Global-Equivariant Homotopy Theory

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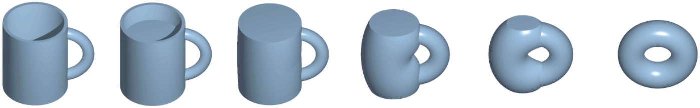
Cohomology theories

Aim of algebraic topology: classify spaces/manifolds up to deformation/homotopy equivalence



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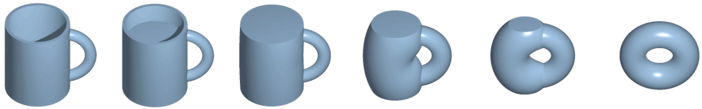

Major tool: cohomology theories

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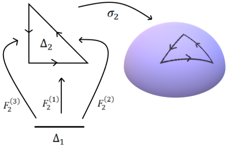

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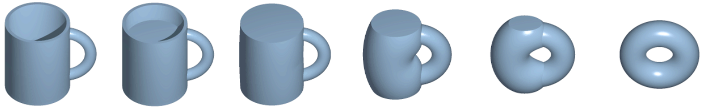

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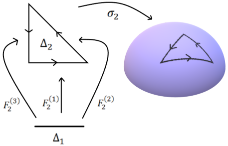
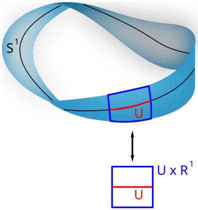


- vector bundles: topological K-theory



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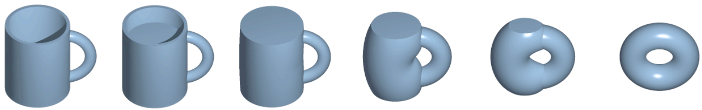

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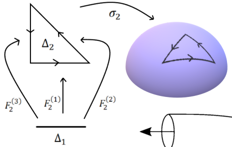

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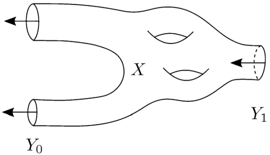

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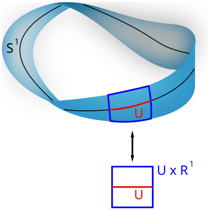

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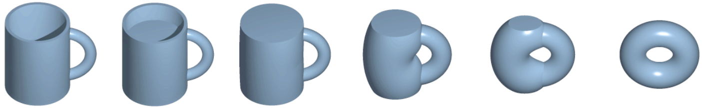

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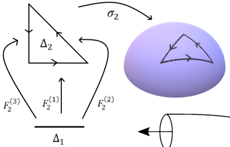

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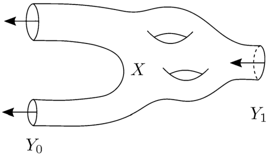

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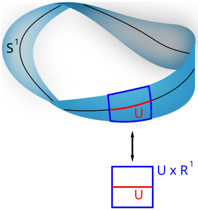
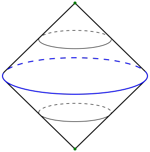


Milestone in algebraic topology (1950/60s): cohomology theories are represented by spectra

Spectra

- informally: force suspension to become an invertible operation



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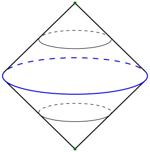

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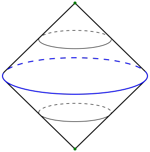

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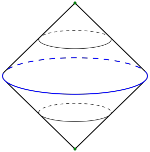

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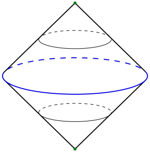

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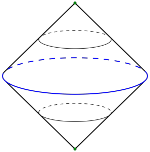

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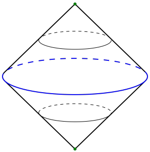

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Advantage: vastly more flexible for manipulating and constructing cohomology theories

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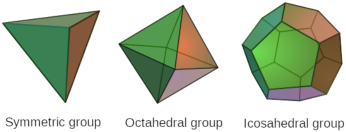

for my story: finite groups
or compact Lie groups G

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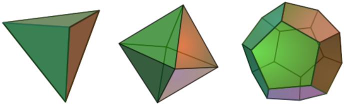

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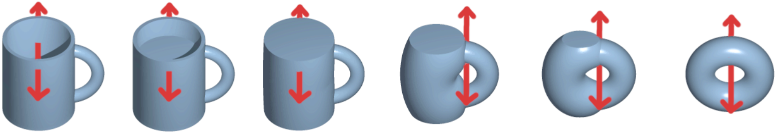

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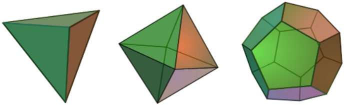

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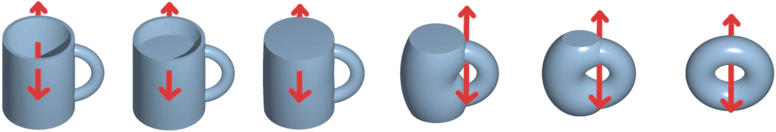

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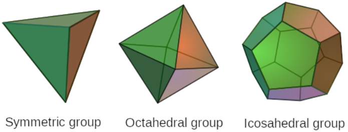

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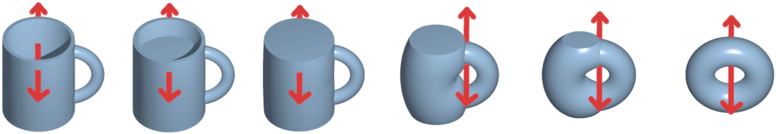

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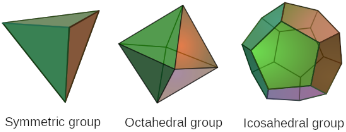

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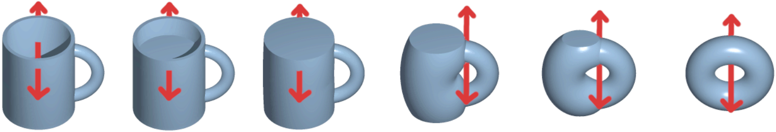

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- G-cohomology theories are represented by G-spectra
- informally: make suspension with all representation spheres invertible, for all G-representations V

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A *G-representation* is a finite-dimensional $\mathbb{R}$-vector space V on which G acts by linear isometries.

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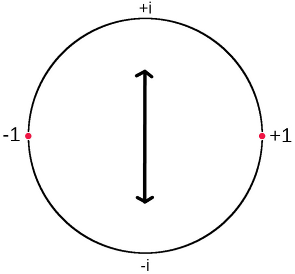

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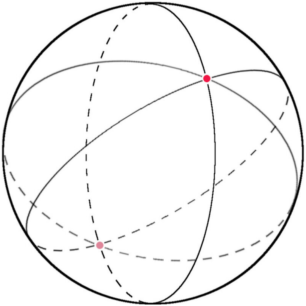

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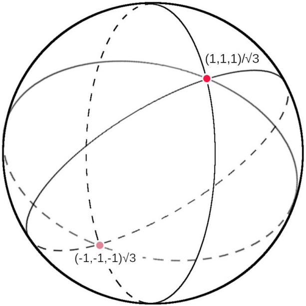

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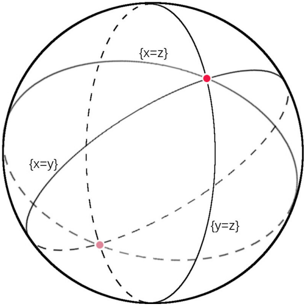

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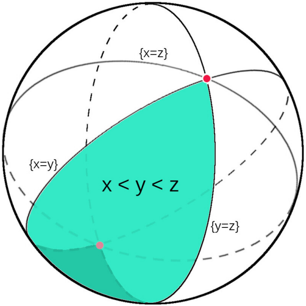

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often several useful equivariant forms of a classical theory; finding the 'best' equivariant form is more an art than a science

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- a general G-Mackey does not extend to a global one

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Global cohomology theories / spectra /	Mackey functors
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- equivariant K-theory / global K-theory spectrum **KU** / complex representation rings — $R(G)$

$$A(G) \xrightarrow{\text{permutation repr'n}} R(G)$$

Examples of global equivariant theories

Global cohomology theories / spectra / **Mackey functors**

- Borel equivariant cohomology /
  global Borel spectrum $b(H\mathbb{Z})$ /
  group cohomology

  $H^k(G; A)$

- equivariant cohomotopy /
  global sphere spectrum $\mathbb{S}$ /
  Burnside rings

  $A(G)$

- equivariant K-theory /
  global K-theory spectrum **KU** /
  complex representation rings

  $R(G)$

- equivariant singular ('Bredon') cohom /
  global Eilenberg-MacLane spectrum $H\underline{\mathbb{Z}}$ /
  constant global Mackey functor

  $\mathbb{Z}$

$$A(G) \xrightarrow{\text{permutation repr'n}} R(G) \xrightarrow{\text{dimension}} \mathbb{Z}$$

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Case study: equivariant bordism

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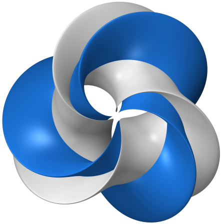

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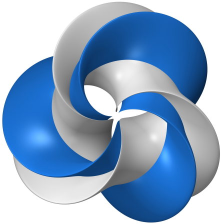

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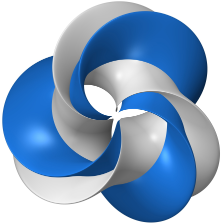

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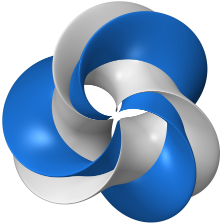

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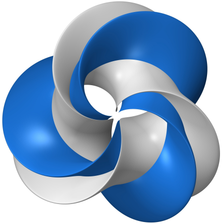

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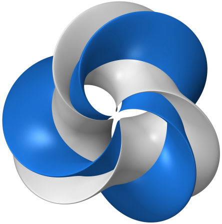

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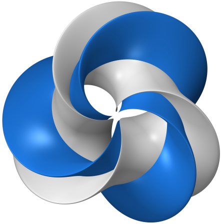

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Variations with equivariant normal structures;
particularly popular: (stably almost) complex structures

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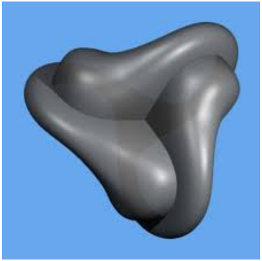
- Alexander (1972): explicit geometric basis constructed from the $\mathbb{R}P^n$'s with involution
  $[x_0 : x_1 : \ldots : x_n] \longmapsto [-x_0 : x_1 : \ldots : x_n]$
- ring structure partially understood



$\mathbb{R}P^2$, involution $=?$

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The theories $\{\mathbf{MU}_G^*\}_{G \text{ cpt Lie}}$ form a global ring spectrum

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inflation/restriction + transfers + double coset formula

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Unexpected features:

- the Chern classes do not generate $\mathbf{MU}^*_{U(n)}$
- some c_k are zero-divisors

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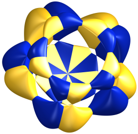

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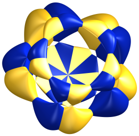

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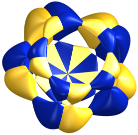

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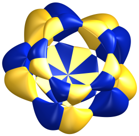

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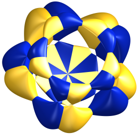

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