Exercises for **Topology I** Sheet 6

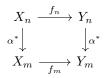
You can obtain up to 10 points per exercise (plus bonus points, where applicable).

Definition. A category C is called *small* if the collection Ob(C) of objects of C forms a set. We write **Cat** for the category whose objects are small categories, with the hom set Hom(C, D) given by the set of functors $C \to D$ (the smallness of C and D guarantees that this is indeed a set). Composition in **Cat** is given by composition of functors.

Exercise 1. 1. Extend the assignment $\mathcal{C} \mapsto \mathcal{C}^{\text{op}}$ to an isomorphism $\mathbf{Cat} \to \mathbf{Cat}$.

- 2. A small category C is called a *groupoid* if every morphism in C is invertible. Show that for every groupoid C there is an isomorphism $C \cong C^{\text{op}}$.
- 3. Give an example of a category \mathcal{C} with $\mathcal{C} \cong \mathcal{C}^{\text{op}}$, but such that \mathcal{C} is *not* a groupoid.

Definition. Let X, Y be simplicial sets. A morphism of simplicial sets $X \to Y$ consists of maps $X_n \to Y_n$ for all $n \ge 0$, such that for every $\alpha \colon [m] \to [n]$ in Δ the diagram



commutes. (In the language of category theory, this means that morphisms of simplicial sets are precisely the *natural transformations* of functors $\Delta^{\text{op}} \rightarrow \text{Set.}$) You can convince yourself that the simplicial sets and morphisms of simplicial sets form a category, with composition given degreewise.

Definition. Let X be a simplicial set. A subsimplicial set of X consists of sets $A_n \subseteq X_n$ for all $n \ge 0$ such that for every $\alpha \colon [m] \to [n]$ in Δ the map $\alpha^* \colon X_n \to X_m$ satisfies $\alpha^*(A_n) \subseteq A_m$.

- **Exercise 2.** 1. Let X be a simplicial sets and let $(A_n \subseteq X_n)_{n\geq 0}$ be a subsimplicial set. Show that the A_n 's can be made into a simplicial set A in a unique way such that the inclusions define a morphism of simplicial sets $A \to X$.
 - 2. For $m, n \ge 0$ and $k \in [n]$ we define

$$(\Lambda_k^n)_m \subseteq (\Delta^n)_m = \{f \colon [m] \to [n] \text{ weakly monotone}\}$$

as the subset of those maps f whose image does not contain $[n] \setminus \{k\}$. Show that the subsets $(\Lambda_k^n)_m$ define a subsimplicial set of Δ^n .

please turn over

Definition. Let X be a set. A partial order on X is a relation \leq satisfying the following conditions:

- 1. Reflexivity: $x \leq x$ for every $x \in X$.
- 2. Transitivity: If $x \leq y$ and $y \leq z$, then also $x \leq z$.
- 3. Antisymmetry: If $x \leq y$ and $y \leq x$, then x = y.

The pair (X, \leq) is called a *partially ordered set*, or *poset* for short. There is a category **Poset** of posets, with morphisms given by the weakly monotone maps.

Exercise 3. 1. Let (X, \leq) be a poset. We define $Ob(\mathcal{C}_X) = X$ and for all $x, y \in X$

$$\operatorname{Hom}_{\mathcal{C}_X}(x,y) = \begin{cases} \{*\} & \text{if } x \leq y \\ \emptyset & \text{otherwise.} \end{cases}$$

Show that there is a unique composition law that turns \mathcal{C}_X into a category.

2. Extend the assignment $X \mapsto \mathcal{C}_X$ to a functor **Poset** \to **Cat** inducing bijections

$$\operatorname{Hom}_{\operatorname{Poset}}(X,Y) \to \operatorname{Hom}_{\operatorname{Cat}}(\mathcal{C}_X,\mathcal{C}_Y).$$

Remark. A functor inducing bijections on hom sets is called *fully faithful*.

3. Let \mathcal{C} be a small category such that $|\operatorname{Hom}_{\mathcal{C}}(x, y)| \leq 1$ for all $x, y \in \operatorname{Ob}(\mathcal{C})$. We define a relation \leq on X by declaring that $x \leq y$ iff there exists a map $x \to y$ in \mathcal{C} . Show that this relation is not a partial order in general, but that it descends to a partial order on the set $\pi_0(\mathcal{C})$ of isomorphism classes of objects.

Exercise 4. Let X be a simplicial set. Given $x, y \in X_0$, we write $x \sim y$ if there exists an element $e \in X_1$ such that $d_1^*(e) = x$ and $d_0^*(e) = y$.

- 1. Show that \sim is reflexive, but in general neither symmetric nor transitive.
- 2. Show that ~ is an equivalence relation provided that every morphism of simplicial sets $f: \Lambda_k^2 \to X$, $0 \le k \le 2$ extends to Δ^2 , i.e. there exists a morphism $F: \Delta^2 \to X$ making the following diagram of simplicial sets commute:



Does the converse also hold?

Hint. First show that morphisms $f: \Delta^n \to X$ of simplicial sets are in bijection with *n*-simplices of X via the assignment $f \mapsto f_n(\operatorname{id}_{[n]})$.

- 3. Let Y be a topological space. Show that ~ defines an equivalence relation on $\mathcal{S}(Y)_0$, and construct a bijection $\mathcal{S}(Y)_0/_{\sim} \cong \pi_0(Y)$.
- *4. (10 bonus points) Let Y be a topological space and let $0 \le k \le n, n \ne 0$. Show that any morphism of simplicial sets $\Lambda_k^n \to \mathcal{S}(Y)$ extends to Δ^n .

Remark. Simplicial sets with this property are called *Kan complexes* (in honor of DANIEL KAN).