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Exercises for **Topology I**
Sheet 4

You can obtain up to 10 points per exercise (plus bonus points, where applicable).

Exercise 1.

1. Let $p\colon E \to B$ be a covering map and let $e \in E$ arbitrary. Show that $p\colon \pi_n(E, e) \to \pi_n(B, p(e))$ is injective for $n = 1$ and bijective for $n \geq 2$.
2. Compute the homotopy groups of $\mathbb{RP}^\infty$.
3. Show that $S^2 \times \mathbb{RP}^\infty$ and $\mathbb{RP}^2$ have isomorphic homotopy groups.

   Remark. We will see later in this course that these two spaces are nevertheless not homotopy equivalent. Thus, the assumption in Whitehead's Theorem that the isomorphism of homotopy groups be induced by a map of spaces is not redundant.

Definition. An *H-space* is a pointed space (X, x_0) equipped with a based map

$$\mu\colon \big(X \times X, (x_0, x_0)\big) \to (X, x_0)$$

such that the two maps $\mu(x_0, -), \mu(-, x_0)\colon X \rightrightarrows X$ are based homotopic to the identity.

Exercise 2.

1. Let $n \geq 0$ and assume the $\mathbb{R}$-vector space $\mathbb{R}^n$ admits the structure of a *unital division algebra*, i.e. there exists an $\mathbb{R}$-bilinear map $\beta\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and an element $e \in \mathbb{R}^n$ such that $\beta(e, -) = \mathrm{id} = \beta(-, e)$ and $\beta(x, y) = 0$ only for $x = 0$ or $y = 0$.

   Show that S^{n-1} admits the structure of an H-space.

   Remark. There are four classically known real unital division algebras: $\mathbb{R}$ itself, $\mathbb{C}$, the quaternions $\mathbb{H}$ (4-dimensional), and the Cayley octonions $\mathbb{O}$ (8-dimensional); $\mathbb{H}$ and $\mathbb{O}$ are non-commutative, and $\mathbb{O}$ is even non-associative. Accordingly, S^0, S^1, S^3 and S^7 all admit H-space structures. By a deep theorem of J. Frank Adams, these are the only spheres with this property, and the four classically known real unital division algebras are the only ones up to isomorphism.

2. Let (X, x_0) be an H-space. Show that $\pi_1(X, x_0)$ is abelian and that for every $n \geq 1$ the map

$$\pi_n(X, x_0) \times \pi_n(X, x_0) \xrightarrow[\cong]{(\mathrm{pr}_{1*}, \mathrm{pr}_{2*})^{-1}} \pi_n\big(X \times X, (x_0, x_0)\big) \xrightarrow{\mu_*} \pi_n(X, x_0)$$

   agrees with the group multiplication.

please turn over

Definition. Let G be a group. We write $\mathbb{Z}[G]$ for the free abelian group with basis G, i.e. any $x \in \mathbb{Z}[G]$ can be uniquely written as

$$x = \sum_{g \in G} n_g g$$

with $n_g \in \mathbb{Z}$ and such that $n_g = 0$ for almost all $g \in G$.

There is then a unique $\mathbb{Z}$-bilinear map $\mathbb{Z}[G] \times \mathbb{Z}[G] \to \mathbb{Z}[G]$ extending the group multiplication on G, and this makes $\mathbb{Z}[G]$ into a ring, called the *(integral) group ring* of G.

Exercise 3.

1. Let A be an abelian group and let G be an arbitrary group, acting on A via group homomorphisms. Show that the action map $G \times A \to A$ extends uniquely to a $\mathbb{Z}$-bilinear map $\mathbb{Z}[G] \times A \to A$, and that this makes A into a $\mathbb{Z}[G]$-module.

2. Let (X, x_0) be a based space and let $n \geq 2$. Show that $\pi_n(X, x_0)$ admits a unique $\mathbb{Z}[\pi_1(X, x_0)]$-module structure such that $[\gamma] \cdot f = \gamma_*(f)$ for every $\gamma\colon ([0,1], \{0,1\}) \to (X, x_0)$ and every $f \in \pi_n(X, x_0)$. Here γ_* denotes the group homomorphism $\pi_n(X, x_0) \to \pi_n(X, x_0)$ constructed in the lecture.

Definition. Let (M_i, m_i), $i = 1, 2, 3$ be pointed sets. A sequence

$$M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3$$

of based maps is called *exact* if $\beta^{-1}(m_3) = \alpha(M_1)$ (i.e. $y \in M_2$ gets mapped to the basepoint iff it has a preimage in M_1).

Exercise 4. Let $f\colon (X, x_0) \to (Y, y_0)$ be any based map. We define

$$\tilde{C}X \coloneqq \big(X \times [0,1]\big) / \big(X \times \{1\} \cup \{x_0\} \times [0,1]\big) \qquad \text{and} \qquad \tilde{C}(f) \coloneqq Y \amalg_{X,i} \tilde{C}X,$$

where $i\colon X \to \tilde{C}X, x \mapsto [x, 0]$. We moreover write $j\colon Y \to \tilde{C}(f)$ for the evident inclusion. We make $\tilde{C}X$ and $\tilde{C}(f)$ into pointed spaces in the unique way for which i and j are based maps.

1. Show that the sequence
$$[\tilde{C}(f), Z]_* \xrightarrow{-\circ j} [Y, Z]_* \xrightarrow{-\circ f} [X, Z]_*$$
of sets of based homotopy classes is exact for every pointed space Z, where the basepoint of $[-, Z]_*$ is chosen to be the class of the constant map in each case.

$*$2. (*5 bonus points*) Assume now that X and Y admit CW-structures with $x_0 \in X_0, y_0 \in Y_0$. Construct a based homotopy equivalence $\tilde{C}(j) \simeq \tilde{\Sigma}X$ for the reduced suspension $\tilde{\Sigma}$ from Sheet 1. Use this to show that we have an infinite sequence
$$\cdots \longrightarrow [\tilde{\Sigma}^2 X, Z]_* \xrightarrow{\partial} [\tilde{C}(\tilde{\Sigma}f), Z]_* \xrightarrow{-\circ j} [\tilde{\Sigma}Y, Z]_* \xrightarrow{-\circ \tilde{\Sigma}f} [\tilde{\Sigma}(X), Z]_* \xrightarrow{\partial} [\tilde{C}(f), Z]_* \xrightarrow{-\circ j} [Y, Z]_* \xrightarrow{-\circ f} [X, Z]_*$$
of pointed sets such that every three term subsequence is exact.