Strata of k-differentials and double ramification cycles

Mittwoch, 29. April 2020 10:01

joint w/ Y. Bae, D. Holmes, R. Pandharipande, R. Schwarz

§1 Two compactifications of loci of K-differentials

Let $g, n \geq 0$ with $2g-2+n>0$.

$$M_{g,n} = \left\{ (C, p_1, \ldots, p_n) \;\middle|\; \begin{array}{l} C \text{ smooth genus } g \text{ curve} \\ p_1, \ldots, p_n \in C \text{ distinct points} \end{array} \right\}$$

⇝ moduli space of smooth curves

⇝ smooth orbifold of dim $3g-3+n$

Let $K \geq 0$, $A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ w/ $\sum a_i = K \cdot (2g-2)$.

$$\mathcal{H}_g^K(A) = \left\{ (C, p_1, \ldots, p_n) \mid \omega_C^{\otimes K} \cong \mathcal{O}_C(\sum a_i p_i) \right\} \subseteq M_{g,n}$$

closed alg. subset

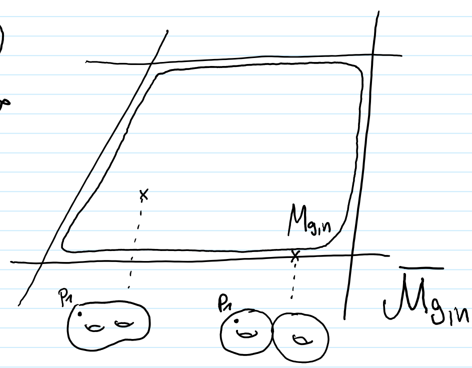
$\Leftrightarrow \exists$ merom. K-diff. η on C with $\mathrm{div}(\eta) = \sum a_i p_i$

<u>Exa</u> $\mathcal{H}_1^K(a, -a) = \left\{ (E, p, q) \mid \mathcal{O}_E \cong \mathcal{O}_E(ap - aq) \right\} \subseteq M_{1,2}$

$= \left\{ (E, p, q) \mid q \text{ non-triv. } a\text{-torsion pt. in } (E, p) \right\}$

<u>Q</u>
- Geometry of $\mathcal{H}_g^K(A)$, e.g. dimension, smoothness, ...
- How to compactify inside the moduli space of stable curves?

$$\overline{M}_{g,n} = \left\{ (C, p_1, \ldots, p_n) \;\middle|\; \begin{array}{l} C \text{ curve of arithm. genus } g, \\ \text{at worst nodal} \\ p_1, \ldots, p_n \in C \text{ distinct smooth pts} \\ \mathrm{Aut}(C, p_1, \ldots, p_n) \text{ finite} \end{array} \right\}$$



- Natural cycle class in $CH^*(\overline{M}_{g,n})$?

<u>A1</u> Closure $\overline{\mathcal{H}_g}(A) \subseteq \overline{M}_{g,n}$ ⇝ strata of K-differentials

→ minimal compactification

→ [Bainbridge - Chen - Gendron - Grushevsky - Möller '16, '16]
Characterization of $(C, p_1, \ldots, p_n) \in \overline{\mathcal{H}}_g^K(A)$
⇝ $\exists$ meromorph. K-diff. on comp. of C, poles & zeros at nodes, K-residue conditions

→ [BCGGM '19, Costantini - Möller - Zachhuber '19]
Construct smooth, modular compactification

$$\mathbb{P}\Xi^K \overline{\mathcal{M}}_{g,n}(A) \longrightarrow \overline{\mathcal{H}}_g^K(A) \subseteq \overline{M}_{g,n}$$

→ [CMZ '20]
- Express the (orbifold) Euler characteristic of strata $\mathcal{H}_g^1(A)$ of differentials in terms of intersection numbers on $\mathbb{P}\Xi^1 \overline{\mathcal{M}}_{g,n}(A)$.
- Calculate them in many examples using computers

<u>Exa</u> $\chi(\mathcal{H}_2^1(2)) = -1/40$.

→ [Sauvaget 20]

Volumes of moduli spaces of flat surfaces ⟷ Intersect. numb. of $[\overline{\mathcal{H}}_g^K(A)]$ w/ natural classes on $\overline{M}_{g,n}$

⇒ Want formula for $[\overline{\mathcal{H}}_g^K(A)]$.

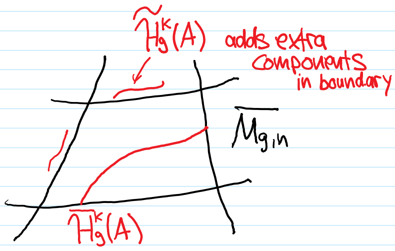
A2 Moduli $\widetilde{\mathcal{H}}_g^K(A) \subseteq \overline{M}_{g,n}$ of twisted K-differentials

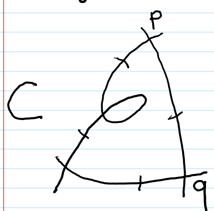
[Farkas–Pandharipande '15]

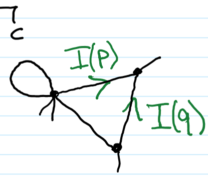
$$\widetilde{\mathcal{H}}_g^K(A) = \{(C, p_1, \dots, p_n) \mid (*)\} \subseteq \overline{M}_{g,n},$$

↑ closed

$$\widetilde{\mathcal{H}}_g^K(A) \cap M_{g,n} = \mathcal{H}_g^K(A)$$





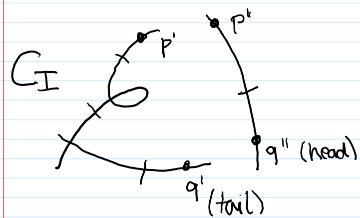


twist I

$I(p), I(q) \in \mathbb{Z}_{>0}$

↳ no strict cycles

↑ ν_I normalizing twisted nodes N_I

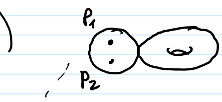


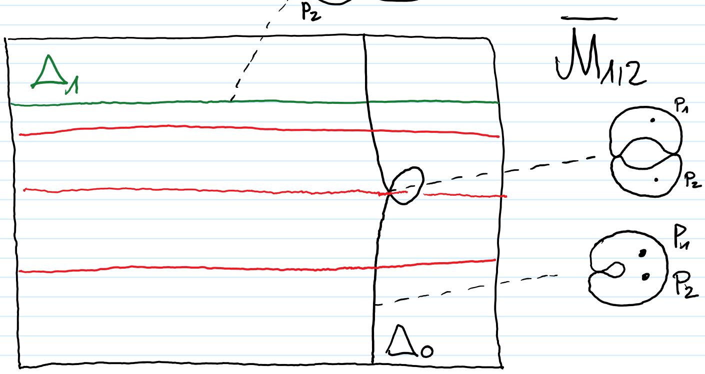
$(*)$: ∃ twist I on Γ_C such that

$$\nu_I^* \omega_C^{\otimes K} \cong \nu_I^* \mathcal{O}_C\left(\sum a_i p_i\right) \otimes \mathcal{O}_{C_I}\left(\sum_{q \in N_I} I(q)(q'' - q')\right)$$

$$\Leftrightarrow \omega_{C_I}^{\otimes K} \cong \mathcal{O}_{C_I}\left(\sum a_i p_i + \sum_{q \in N_I} (-I(q)-K)q' + (I(q)-K)q''\right)$$

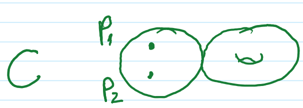
Exa $\widetilde{\mathcal{H}}_1^K(a,-a)$

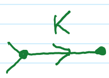




$$\widetilde{\mathcal{H}}_1^K(a,-a) = \overline{\mathcal{H}}_1^K(a,-a) \cup \Delta_1$$

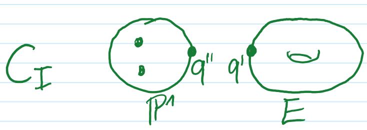
↓





(Γ, I)

↑ ν_I



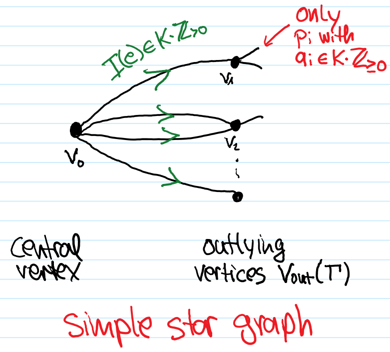
→ $\omega_{\mathbb{P}^1}^{\otimes K} \cong \mathcal{O}_{\mathbb{P}^1}(a p_1 - a p_2 - (K+K) q'')$

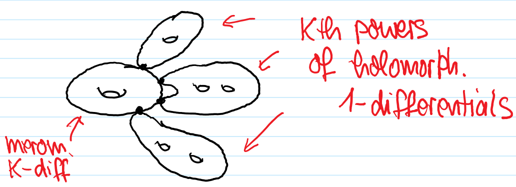
→ $\omega_E^{\otimes K} \cong \mathcal{O}_E((K-K) q')$

Thm (FP'15, S'16) $k \geq 1$

$Z \subset \widetilde{H}^k_g(A)$ component

Generic Γ', I | **Generic $(C, p_1, \ldots, p_n)$**





$\Rightarrow$ Components of $\widetilde{H}^k_g(A)$ supported in boundary of $\overline{M}_{g,n}$ are parameterized by products of spaces $\widetilde{H}^{k_i}_{g(v_i)}(A_i)$

$\rightsquigarrow$ Motivation for definition?

$(C, p_1, \ldots, p_n) \mapsto (C, \omega_C^{\otimes k}(-\sum a_i p_i))$

$\sigma_{A,k} : \overline{M}_{g,n} \to \mathrm{Pic}_{g,0} = \{(C, \mathcal{L})\}$ (line bdle on C total deg. 0)

$\pi : \mathrm{Pic}_{g,0} \to \mathfrak{M}_g$, section e: $\mathcal{L} = \mathcal{O}_C$

$\overline{M}_{g,n} \longrightarrow \mathfrak{M}_g = \{C\}$ (prestable genus g)

Prop (BHPSS'20)
$$\widetilde{H}^k_g(A) = \sigma_{A,k}^{-1}(\overline{e})$$
($\overline{e}$: closure of image of e in $\mathrm{Pic}_{g,0}$)

$\rightsquigarrow$ $\overline{e} \subset \mathrm{Pic}_{g,0}$ has pure codim g; what about $\widetilde{H}^k_g(A)$?

§2 Dimension theory & weighted fundamental class

Thm (F-P'15 ($k=1$), S'16 ($k>1$))
For $k \geq 1$, $\widetilde{H}^k_g(A)$ has pure codim g in $\overline{M}_{g,n}$, <u>except</u> if $A = k \cdot A'$ for $A' \in \mathbb{Z}^n_{\geq 0}$, in which case
$$\overline{H}^1_g(A') \subseteq \widetilde{H}^k_g(A)$$
is a union of comp. of codim $g-1$.

<u>Idea of Pf</u> over $M_{g,n}$: $\sigma_{A,k}$ and e meet transversally (Deformation theory) $\Rightarrow$ $H^k_g(A) \subseteq M_{g,n}$ smooth

in $\partial \overline{M}_{g,n}$: recursive argument $\square$

$\rightsquigarrow$ What about cycle theory?

Conjecture A (Janda–Pandharipande–Pixton–Zvonkine '15 ($k=1$), S '16 ($k>1$))

Let $k \geq 1$ and $A \neq kA'$ for $A' \in \mathbb{Z}^n_{\geq 0}$. ⇝ $\widetilde{\mathcal{H}}^k_g(A)$ pure codim g

Then

$$\sum_{\substack{Z \text{ component} \\ \text{of } \widetilde{\mathcal{H}}^k_g(A)}} m_Z \cdot [Z] = 2^{-g} P_g^{g,k}(\tilde{A}) \in CH^g(\overline{\mathcal{M}}_{g,n}) \quad (\star)$$

m_Z: explicit pos. integer

$\tilde{A} = (a_1+k, \dots, a_n+k)$

$P_g^{g,k}$: Pixton's formula for double ramification (DR) cycle in the tautological ring $R^*(\overline{\mathcal{M}}_{g,n}) \subseteq CH^*(\overline{\mathcal{M}}_{g,n})$

R^*: explicit generators, formul. for products, ...

Proof ([HS'19, BHPSS'20])

Recall

$$\begin{array}{ccc} & & \mathfrak{Pic}_{g,0} \\ & \overset{\sigma_{A,k}}{\nearrow} & \pi\downarrow \uparrow e \\ \overline{\mathcal{M}}_{g,n} & \longrightarrow & \mathfrak{M}_g \end{array}$$

$\widetilde{\mathcal{H}}^k_g(A) = \sigma_{A,k}^{-1}(\bar{e})$

→ [HS'19] $\left(\begin{array}{l}\text{Intersection multiplicity} \\ \text{of } \sigma_{A,k} \text{ and } \bar{e} \text{ along} \\ Z \subset \widetilde{\mathcal{H}}^k_g(A)\end{array}\right) = m_Z$ ⇒ LHS of $(\star) = \sigma_{A,k}^*([\bar{e}])$

→ [BHPSS'20] Show Pixton style formula $[\bar{e}] = P_g^g \in CH^g(\mathfrak{Pic}_{g,0})$

$\Rightarrow$ (||) RHS of $(\star) = \sigma_{A,k}^*(P_g^g)$ (short computation)

□

⇒ $[\bar{e}] \in CH^g(\mathfrak{Pic}_{g,0})$ is the universal twisted DR cycle

⇝ all classical DR cycles are pullbacks under $\sigma_{A,k}: \overline{\mathcal{M}}_{g,n} \to \mathfrak{Pic}_{g,0}$

⇝ What about cycles $[\overline{\mathcal{H}}^k_g(A)]$?

Conj. A $[\overline{\mathcal{H}}^k_g(A)] + \left(\begin{array}{l}\text{boundary comp.} \\ \text{of } \overline{\mathcal{H}}^k_g(A)\end{array}\right) = \left(\begin{array}{l}\text{explicit} \\ \text{formula}\end{array}\right)$

boundary comp.: parameterized by smaller-dim'l spaces $\overline{\mathcal{H}}^{k_i}_{g(v_i)}(A_i)$

⇒ we can set up recursion.

⇒ recursive formula for $[\overline{\mathcal{H}}^k_g(A)]$.

I owe you

→ discussion of taut. classes & Pixton's formula

→ proof that $[\bar{e}] = P_g^g \in CH^g(\mathfrak{Pic}_{g,0})$

Pixton's formula on the Picard stack

Freitag, 10. April 2020 11:36

§3 Chow group of $\mathfrak{Pic}_{g,0}$

$\rightsquigarrow$ use operational / bivariant / Chow cohom. approach (Fulton chap. 17)

Let S be finite type scheme

$$S \xrightarrow{\varphi} \mathfrak{Pic}_{g,0} \overset{\text{Def.}}{\longleftrightarrow} \begin{array}{c} C \;\; \mathcal{L} \\ \downarrow \pi_S \\ S \end{array}$$ family of curves + line bundle

An operat. class $\alpha \in CH^c_{op}(\mathfrak{Pic}_{g,0})$ is data of

$$\Big(\alpha(\varphi) : CH_*(S) \longrightarrow CH_{*-c}(S) \Big)_{\varphi : S \to \mathfrak{Pic}_{g,0}}$$

$$\beta \longmapsto (\varphi^* \alpha) \cap \beta$$ ↖ all such morph.

Compatible with prop. pushforw., flat pullback, Gysin pullback

With some work:

$$\overline{\mathcal{E}} \subset \mathfrak{Pic}_{g,0} \rightsquigarrow [\overline{\mathcal{E}}] \in CH^g_{op}(\mathfrak{Pic}_{g,0})$$

($\overline{\mathcal{E}}$ closed, pure codim g)

"Poincaré dual of fund class"

§4 Tautological classes on $\mathfrak{Pic}_{g,0}$

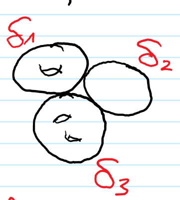
<u>Idea</u>
- Define $R^*(\mathfrak{Pic}_{g,0}) \subseteq CH^*_{op}(\mathfrak{Pic}_{g,0})$
- Express $[\overline{\mathcal{E}}]$ as elem. in $R^*(\mathfrak{Pic}_{g,0})$

- $$\begin{array}{c} \mathcal{C} \leftarrow \mathcal{L} \\ \downarrow F \\ \mathfrak{Pic}_{g,0} \end{array}$$ ← ∃ universal line bundle!

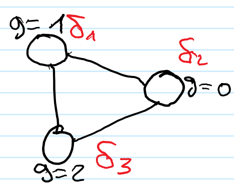
  $\rightsquigarrow \eta := F_*(c_1(\mathcal{L})^2) \in CH^1_{op}(\mathfrak{Pic}_{g,0})$

- boundary strata of $\mathfrak{Pic}_{g,0}$ $\longleftrightarrow$ $\left.\begin{array}{l}\text{prestable graphs } \Gamma \\ + \text{degree fct. } \delta : V(\Gamma) \to \mathbb{Z}\end{array}\right\} \Gamma_\delta$

  $\sum \delta_i = 0$



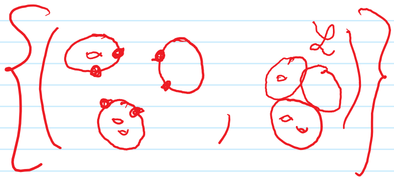
deg $\mathcal{L}$



- Given T_δ' have gluing morphism

$$j_{T_\delta'} : \mathrm{Pic}_{T_\delta'} \longrightarrow \mathrm{Pic}_{g,0}$$
$$\downarrow \pi_{T_\delta'}$$
$$\prod_{v \in V(T')} \mathrm{Pic}_{g(v), n(v), \delta(v)}$$

$\mathrm{Pic}_{T_\delta'} =$



- $\mathrm{Pic}_{g,n,d} = \{(C, p_1, \dots, p_n, \mathcal{L})\}$

  $\mathbb{L}_i \longrightarrow \mathrm{Pic}_{g,n,d}$ line bundle, $\mathbb{L}_i|_{(C,p_1,\dots,p_n,\mathcal{L})} = T^*_{p_i} C$

  $\rightsquigarrow \psi_i = c_1(\mathbb{L}_i) \in CH^1_{op}(\mathrm{Pic}_{g,n,d})$ $\rightsquigarrow$ via $\pi^*_{T_\delta'}$: also in $CH^1(\mathrm{Pic}_{T_\delta'})$

Pixton's formula (shape)

$$P_g^g = \sum_{T_\delta', w, c} \eta^c \, (j_{T_\delta'})_* \left(\text{polynomial in } \psi\text{-classes on } \mathrm{Pic}_{T_\delta'} \right)$$

Thm (BHPSS '20)

We have $[\bar{e}] = P_g^g \in CH^g_{op}(\mathrm{Pic}_{g,0})$.

§5 DR cycles with targets

[JPPZ] X nonsing. proj. variety, $\mathcal{L} \to X$ line bundle

Given $\beta \in H_2(X, \mathbb{Z})$:

$$\overline{M}_{g,n,\beta}(X) = \{ (C, p_1, \dots, p_n) \xrightarrow{f} X \}$$

stable maps of degree β ← every comp. of C not contract. by f is stable.

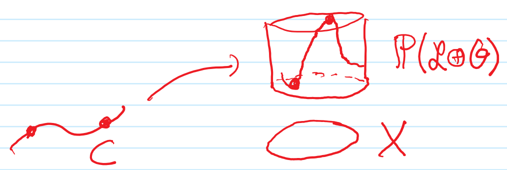
↑ prestable

Given $A = (a_1, \dots, a_n) \in \mathbb{Z}^n$ w/ $\sum a_i = \int_\beta c_1(\mathcal{L})$, the paper [JPPZ] defines a DR cycle $DR_{g,A,\beta}(X, \mathcal{L})$ on $\overline{M}_{g,n,\beta}(X)$ compactifying the condition

$$f^* \mathcal{L} \cong \mathcal{O}_C\left(\sum a_i p_i\right)$$

↖ idea use moduli space of maps to $\mathbb{P}(\mathcal{L} \oplus \mathcal{O}) \to X$

They show a Pixton-style formula $P^g_{g,A,\beta}(X, \mathcal{L})$ for $DR_{g,A,\beta}(X, \mathcal{L})$ ← use localization by $\mathbb{C}^*$-act. on $\mathbb{P}(\mathcal{L} \oplus \mathcal{O})$



What we can show:

$$\varphi: \overline{M}_{g,n,\beta}(X) \longrightarrow \mathfrak{Pic}_{g,0}$$

$$((C,p_1,\dots,p_n) \xrightarrow{f} X) \longmapsto (C, f^*\mathcal{L}(-\sum a_i p_i))$$

$\overset{\text{Thm}}{\Rightarrow}$ $\varphi^*([\bar{e}]) \cap [\overline{M}_{g,n,\beta}(X)]^{vir} = DR_{g,A,\beta}(X,\mathcal{L})$

$\varphi^*(P_g^g) \cap$ - - - - - $= P^g_{g,A,\beta}(X,\mathcal{L})$))[JPPZ]

Idea of Proof of main Theorem

For $X = \mathbb{P}^n$, $\beta = d\cdot[L]$ we can use the maps φ above as "charts" of $\mathfrak{Pic}_{g,0}$

↳ Known equal. from [JPPZ] $\Rightarrow$ $[\bar{e}]$, P_g^g act in same way on $[-]^{vir}$ via φ

↳ for $n,d \gg 0$, there is large open subs. of $\overline{M}_{g,n,\beta}(X)$ on which virt. fund. class = usual fund. class

↳ verify that knowing act. of $[\bar{e}]$, P_g^g on these is enough to show equality in $CH^g_{op}(\mathfrak{Pic}_{g,0})$. #