Dr. E. Hellmann WS 2013/14

Algebraic Geometry I

Exercise Sheet 8

Due Date: 12.12.2013

Exercise 1:

(i) Let $(X, \mathcal{O}_X)$ be a scheme and $Z \subset X$ be a (locally) closed subset. Show that there is a unique structure of a closed subscheme $(Z, \mathcal{O}_Z)$ on Z such that $(Z, \mathcal{O}_Z)$ is reduced.

If we choose $Z = X$ in (i), then we write X_{red} for the resulting scheme and call it the *reduced scheme underlying* X.

(ii) Let $f : X \to Y$ be a morphism of schemes. Show that f induces a morphism $f_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$ such that the diagram

$$\begin{array}{ccc} X_{\mathrm{red}} & \xrightarrow{f_{\mathrm{red}}} & Y_{\mathrm{red}} \\ {\scriptstyle \iota_X}\downarrow & & \downarrow{\scriptstyle \iota_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

commutes, where $\iota_X : X_{\mathrm{red}} \hookrightarrow X$ and $\iota_Y : Y_{\mathrm{red}} \hookrightarrow Y$ are the canonical closed immersions.

Exercise 2:

(i) Let $f : X \to Y$ be a morphism of schemes and let $j : Z \hookrightarrow Y$ be an immersion. Then f factors over j if and only if $f(X) \subset j(Z)$ and $\mathcal{O}_Y \to f_*\mathcal{O}_X$ factors over $\mathcal{O}_Y \to j_*\mathcal{O}_Z$.

(ii) Let $f : X \to Y$ be morphism of schemes and let $Z \subset Y$ be a subscheme. Assume in addition that X is reduced. Show that f factors over the inclusion $Z \hookrightarrow Y$ if and only if $f(X) \subset Z$.

Exercise 3:

Let k be a field. Describe the fibers of the following morphisms. Which fibers are reduced? Which fibers are irreducible? Try to draw pictures of the situation.

(i) $\operatorname{Spec} k[X,Y]/(XY-1) \to \operatorname{Spec} k[X]$

(ii) $\operatorname{Spec} k[X,Y]/(X^2-Y^2) \to \operatorname{Spec} k[X]$

(iii) $\operatorname{Spec} k[X,Y]/(X^2+Y^2) \to \operatorname{Spec} k[X]$

(iv) $\operatorname{Spec} k[X,Y]/(XY) \to \operatorname{Spec} k[X]$

(v) $\operatorname{Spec} k[X,Y,Z]/(Y^2-XZ) \to \operatorname{Spec} k[X]$

(vi) $\operatorname{Spec} k[T,U,V,W]/((U+T)W, (U+T)(U^3+U^2+UV^2-V^2)) \to \operatorname{Spec} k[T]$

(In all cases the morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is induced by the obvious ringhomomorphism $A \to B$.)

Exercise 4:

(i) Let K be a field and let $\bar{K}$ denote an algebraic closure of K. Let L be an algebraic extension of K. Compute $\operatorname{Spec} L \times_{\operatorname{Spec} K} \operatorname{Spec} \bar{K}$.

(ii) Let R be a ring. Show that $\mathbb{A}^n_R \times_{\operatorname{Spec} R} \mathbb{A}^m_R \cong \mathbb{A}^{n+m}_R$ and $\mathbb{A}^n_R \times_{\operatorname{Spec} R} \operatorname{Spec} S \cong \mathbb{A}^n_S$ for all R-algebras S.

(iii) Let k be an algebraically closed field. Show that the product $\mathbb{P}^1_k \times_{\operatorname{Spec} k} \mathbb{P}^1_k$ embeds into $\mathbb{P}^3_k$ as a closed subscheme which is isomorphic to the quadric $V_+(T_0T_3 - T_1T_2)$.

(Hint: Show that in homogenous coordinates (on closed points) the immersion is given by $(x_0 : x_1), (y_0 : y_1) \mapsto (x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1)$ and use an affine cover of $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$.)

Homepage: `www.math.uni-bonn.de/people/hellmann/alggeom`