Dr. E. Hellmann WS 2013/14

Algebraic Geometry I

Exercise Sheet 11

Due Date: 16.01.2014

Exercise 1:

Let X be a scheme and $U \subset X$ be an open subscheme. Let us write $j : U \to X$ for the canonical inclusion and let $\mathscr{F}$ be an $\mathcal{O}_U$-module. Define the sheaf $j_!\mathscr{F}$ on X to be the sheafification of

$$j_!\mathscr{F} : V \longmapsto \begin{cases} 0 & V \not\subset U \\ \mathscr{F}(V) & V \subset U \end{cases}$$

(i) Show that $j_!\mathscr{F}$ is an $\mathcal{O}_X$-module.

(ii) Given $x \in X$ show that $(j_!\mathscr{F})_x = \mathscr{F}_x$ if $x \in U$ and $(j_!\mathscr{F})_x = 0$ otherwise.

(iii) Show that $j^*j_!\mathscr{F} = \mathscr{F}$.

(iv) Use the functor $j_!$ to give examples of $\mathcal{O}_X$ modules that are not quasi-coherent.

Exercise 2:

Let $X = \operatorname{Spec} A$ be an affine scheme. Show that the functors $\tilde{(-)} : (A{-}\mathrm{Mod}) \to (\mathcal{O}_X{-}\mathrm{mod})$ and $\Gamma(X, -) : (\mathcal{O}_X{-}\mathrm{mod}) \to (A{-}\mathrm{mod})$ are adjoint, i.e. for an A-module M and an $\mathcal{O}_X$-module $\mathscr{F}$ there is an isomorphism

$$\mathrm{Hom}_A(M, \Gamma(X, \mathscr{F})) \cong \mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathscr{F})$$

which is functorial in M and $\mathscr{F}$.

Exercise 3:

(i) Let $f : X \to Y$ be a morphism of schemes such that

$$\begin{aligned} &f \text{ is quasi-compact and for } W \subset Y \text{ open and quasi-compact} \\ &U, V \subset f^{-1}(W) \text{ quasi-compact open} \Longrightarrow U \cap V \text{ quasi-compact open.} \end{aligned} \tag{1}$$

Let $\mathscr{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Show that $f_*\mathscr{F}$ is a quasi-coherent $\mathcal{O}_Y$-module. (Hint: Reduce to the case $Y = \operatorname{Spec} B$ and find a finite cover $X = \bigcup U_i$ with open affine subschemes U_i and finite open affine covers $U_i \cap U_j = \bigcup_{k \in I_{ij}} V_{ijk}$. Then use the fact that $f_*(\mathscr{F}|_{U_i})$ and $f_*(\mathscr{F}|_{V_{ijk}})$ are known to be quasi-coherent.)

(ii) Let $f : X \to Y$ be a morphism on schemes such that either X is noetherian (as a topological space) and on underlying topological spaces f is the embedding of an open subspace, or f is the embedding of a closed subspace. Show that f satisfies the property (1).

Exercise 4:

Let k be an (algebraically closed) field and fix an k-rational point $\infty \in \mathbb{P}^1_k = X$. Define sheaves of $\mathcal{O}_X$-modules $\mathcal{O}(1)$ and $\mathcal{O}(-1)$ by

$$\mathcal{O}(-1) : U \longmapsto \{f \in \Gamma(U, \mathcal{O}_X) \mid f(\infty) = 0\}$$

and $\mathcal{O}(1) = \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}(-1), \mathcal{O}_X)$. Further for an integer $n \geq 0$ we define $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$ and $\mathcal{O}(-n) = \mathcal{O}(-1)^{\otimes n}$. Let $n \in \mathbb{Z}$.

(i) Show that $\mathcal{O}(n)$ is a quasi-coherent $\mathcal{O}_X$-module.

(ii) Show that for $m, n \in \mathbb{Z}$ one has $\mathcal{O}(m) \otimes_{\mathcal{O}_X} \mathcal{O}(n) \cong \mathcal{O}(m+n)$.

(iii) Show that $\Gamma(X, \mathcal{O}(n))$ is a finite dimensional k-vector space of dimension

$$\dim_k \Gamma(X, \mathcal{O}(n)) = \begin{cases} 0 & n < 0 \\ n+1 & n \geq 0 \end{cases}$$

(iv) Show that the functor $\Gamma(X, -)$ on the category of quasi-coherent $\mathcal{O}_X$-modules does not commute with tensor products. This is find quasi-coherent $\mathcal{O}_X$-modules $\mathscr{F}$ and $\mathscr{G}$ such that

$$\Gamma(X, \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}) \ncong \Gamma(X, \mathscr{F}) \otimes_{\Gamma(X, \mathcal{O}_X)} \Gamma(X, \mathscr{G}).$$

(v) Give an interpretation of $\mathcal{O}(n)$ as a subsheaf of the constant sheaf $\underline{K}$ on X, where $K = k(\mathbb{P}^1_k)$ denotes the function field of $\mathbb{P}^1_k$, i.e. $K = \mathcal{O}_{X,\eta} = \operatorname{Frac} \mathcal{O}_{X,x}$, where η is the generic point of X and x is an arbitrary point of X

Homepage: `www.math.uni-bonn.de/people/hellmann/alggeom`