S4D2 – Graduate Seminar on Topology – Morse Theory
S2D3 – Hauptseminar Differentialtopologie – Morse-Theorie

Martin Palmer-Anghel // Sommersemester 2018 // Tuesdays 12:15–14:00, seminar room 1.008

Summary. The aims of this seminar are the following.

- Introduce an important and often-used technique in differential topology: Morse theory. The idea is that, given a sufficiently nice function f on a manifold M, the topology of M is "concentrated" at the singular (critical) points of f, and one can extract information about the manifold by studying these singularities.
- Use this theory to prove the *h-cobordism theorem*, which states that any simply-connected h-cobordism of dimension at least 6 is trivial (of the form $M \times [0,1]$). An easy corollary of this theorem is the (topological) Poincaré conjecture in high dimensions. Another corollary is the high-dimensional smooth Schoenflies theorem.
- The rough idea of the proof is that any cobordism admits a Morse function with finitely many critical points. If it has *no* critical points, then it is trivial. The proof therefore consists in modifying a given Morse function to eliminate all of its critical points, cancelling them in pairs, until none remain. The key technical step is the cancellation of pairs of critical points using the so-called Whitney trick.
- At the end we will also briefly discuss the *smooth* Poincaré conjecture in high dimensions — which is *false*. Since the topological PC is true and the smooth PC is false in high dimensions, there are non-standard ("exotic") smooth structures on S^n for large n (starting with $n = 7$).

The aims of the seminar are quite ambitious, but the h-cobordism theorem is a high point of classical differential topology, so it should be worth the effort for those who like this subject.

Preparing your talk. The ideas of most steps of the proof are in many places very geometric, so you should try to explain the underlying ideas with pictures, wherever possible. The proofs also become quite detailed, so part of the preparation for your talk will be to choose a good path through the material that your talk covers. There should be a balance between (a) keeping an eye on the big picture (not becoming too lost in the details) and (b) explaining all of the necessary ideas, without skipping the more difficult, technical results that are needed for the proof. Your talk should have a clear overall structure.

You should discuss the topic of your talk with me at the latest two weeks in advance, by which time your should already have an overall plan for your talk. My office hours will be fixed later (see the webpage for the seminar).

References. The main reference defines what your talk should be about. The other references are there in case they help you to prepare, but they are entirely optional (especially for the first few talks, where there are many references).

Your talk should be 90 minutes, allowing time for questions in between (so plan for $\sim$75 minutes).

Webpage. www.math.uni-bonn.de/people/palmer/Morse.html

Background for the seminar.

- **Algebraic topology:** The lecture course *Topologie I*, Wintersemester 17/18.
- Additionally, we will use a little bit of the course *Topologie II*, Sommersemester 18 (namely cohomology and Poincaré duality).
  Textbook reference: [A. Hatcher, *Algebraic topology*, (link to pdf)].
- Some basic concepts from differential topology. The course *Global Analysis I*, Wintersemester 17/18 would for example be useful to have attended, but it is certainly not a critical prerequisite.
  (Here are the lecture notes + additional notes on flows for this course.)
  Textbook references: the books
  - J. M. Lee, *Introduction to smooth manifolds.*
  - J. M. Lee, *Riemannian manifolds – an introduction to curvature.*

  were the recommended literature for *Global Analysis I*. Also recommended are:
  - V. Guillemin, A. Pollack, *Differential topology.*
  - M. Hirsch, *Differential topology.*
  - A. Kosinski, *Differential manifolds.*

Abstracts for the talks.

1. Morse functions. (10.04) (*Speaker:* **Lukas Bonfert**)
The definition and local structure of Morse functions on smooth manifolds, more generally manifold triads. Existence (and examples) of Morse functions. The *Morse number* of a manifold triad.
Main references. [M^h, §2 and part of §1], [M^m, §2]
Other references. [Mat, §2.2], [Ko, §IV.3–4] [Nic, §1.1 and §1.2], [AD, §1]

2. Cobordisms I. (17.04) (*Speaker:* **Jonas Cremer**)
Cobordisms, gradient-like vector fields, cobordisms of Morse number 0, subadditivity of the Morse number of cobordisms.
Main reference. [M^h, §1 and §3 up to page 26]
Other references. [M^m, §3], [Mat, §2.3], [Ko, §I.7] [Nic, §2.2], [AD, §2.1]

3. Cobordisms II. (24.04) (*Speaker:* **Branko Juran**)
Surgery, elementary cobordisms, the fundamental theorem of Morse theory (Theorems 3.12 and 3.13 of [M^h]), existence of handle decompositions.
Main references. [M^h, §3 pages 27–36], [Mat, Theorem 3.4]
Other references. [M^m, §3], [Mat, §3.1], [Ko, §VI.9 and §VII.1–2] [Nic, §2.2], [AD, §2.1]

4. Rearrangement of cobordisms (a.k.a. handle sliding). (08.05)
(*Speaker:* **Christian Nöbel**)
Modification of gradient-like vector fields for a given Morse function. Modifying a Morse function so that it is self-indexing.
Main reference. [M^h, §4]
Other references. [Mat, §3.3], [Nic, Theorem 2.4.11], [AD, §2.2]

5. Morse homology. (15.05) (*Speaker:* **Robin Stoll**)
The Reeb sphere theorem. Morse homology, with two applications: the Morse inequalities and Poincaré-Lefschetz duality.
Main references. [M^m, §4], [M^h, Lemma 6.3 on page 69 plus §7 up to page 92], [Ko, §VII.4]
Other references. [Mat, §4.2], [Ko, §VI.10, §VII.3, §VII.5], [Nic, §2.3], [Hut, §2–3], [AD, §3–4]

6. Cancellation of critical points I. (29.05) (*Speaker:* **Niklas Hellmer**)
Another lemma about modifying gradient-like vector fields for a given Morse function. Proof of Theorem 5.4 (the First Cancellation Theorem), *assuming* Assertion 6 on page 55.
Main reference. [M^h, §5 up to page 55]
Other references. [Mat, §3.4, in particular Theorem 3.28]

7. Cancellation of critical points II. (05.06) (*Speaker:* **Carsten Uhlig**)
Proof of Assertion 6, via the technical Theorem 5.6 about self-embeddings of Euclidean spaces.
Main reference. [M^{h}, §5 pages 56–66]

8. Cancellation of critical points III (the Whitney trick I). (12.06)
(*Speaker:* **Jonas Antor**)
Proof of Theorem 6.4 (the Second Cancellation Theorem) — a generalisation of the First Cancellation Theorem, weakening the assumption of *geometric* intersection in a single point to *algebraic* intersection number equal to ± 1 — *assuming* Theorem 6.6 (which is the Whitney trick). Then a proof of Theorem 6.6, *assuming* Lemma 6.7.
Main reference. [M^{h}, §6 up to page 74]
Other references. [Sc, §1.5 and pages 54–57]

9. Cancellation of critical points IV (the Whitney trick II). (19.06)
(*Speaker:* **Lennart Ronge**)
Proof of Lemma 6.7, completing the proof of the Whitney trick and the Second Cancellation Theorem.
Main reference. [M^{h}, §6 pages 73–84]
Other references. [Sc, §1.5 and pages 54–57]

10. Elimination of critical points in the middle dimensions. (26.06)
(*Speaker:* **Ferdinand Wagner**)
Proof the Basis Theorem (7.6). Then prove that, on a simply-connected h-cobordism of dimension $n \geqslant 6$, all critical points of index $\neq 0, 1, n-1, n$ may be eliminated (in pairs) (Theorem 7.8).
Main reference. [M^{h}, §7 pages 92–99]
Other references. [Sc, §1.6]

11. Elimination of critical points in the extremal dimensions. (03.07)
(*Speaker:* **Lars Munser**)
Prove that critical points of index $0, 1, n-1, n$ may also be eliminated in pairs, via the creation of auxiliary critical points (Theorem 8.1). Deduce the h-cobordism theorem (Theorem 9.1), as well as Theorem 9.2.
Main reference. [M^{h}, §8 and pages 107–108]
Other references. [Sc, §1.6], [Ko, §VIII.3]

12. The Poincaré conjecture and the Schoenfließ theorem. (10.07)
(*Speaker:* **Dídac Violan Arís**)
Prove Propositions A, B, C and D in §9 of [M^{h}], using the h-cobordism theorem. (State without proof the additional results of Kervaire-Milnor, Cerf/Palais and Brown that are also needed.)
Main reference. [M^{h}, §9]

13. Introduction to exotic spheres. (17.07)
(*Speakers:* **Xiaowen Dong** and **Tobias Fleckenstein**)
Introduce the group Θ_n of *homotopy n-spheres*, and show that it is an abelian group. Explain the connection to the smooth Poincaré conjecture. Give a brief survey of what is known about the group Θ_n, including the facts that (a) it is often *non-zero* (starting from $n = 7$) but on the other hand (b) it is *finite*.
Main reference. [KM]
Other references. [Le], [Ko, §X], [M^7]

References.

- [$\mathrm{M^h}$] = **J. Milnor, *Lectures on the h-cobordism theorem.*** (Main reference for the seminar.)

- [AD] = M. Audin, M. Damian, *Théorie de Morse et homologie de Floer.*
- [Hut] = M. Hutchings, *Lecture notes on Morse homology.*
- [KM] = M. Kervaire, J. Milnor, *Groups of homotopy spheres.*
- [Ko] = A. Kosinski, *Differential manifolds.*
- [Le] = J. P. Levine, *Lectures on groups of homotopy spheres.*
- [Mat] = Y. Matsumoto, *An introduction to Morse theory.*
- [$\mathrm{M^7}$] = J. Milnor, *On manifolds homeomorphic to the 7-sphere.*
- [$\mathrm{M^m}$] = J. Milnor, *Morse Theory.*
- [Nic] = L. Nicolaescu, *An invitation to Morse theory.*
- [Sc] = A. Scorpan, *The wild world of 4-manifolds.*