Summer term 2024 Dr. Andreas Mihatsch
Elliptic curves and their moduli spaces

Homework problems (due April 24)

Problem 1

Let k be a field and A a (not necessarily commutative) finite-dimensional k-algebra.

(a) Prove that the functor $\underline{A}$ on k-schemes given by

$$\underline{A} = \mathcal{O}_T(T) \otimes_k A$$

is representable by $\mathbb{A}^n_k$, where $n = \dim_k(A)$. Show that there also exists an affine scheme that represents the functor

$$\underline{A}^\times(T) = (\mathcal{O}_T(T) \otimes_k A)^\times\,. \tag{1}$$

(b) Define a group scheme structure on $\underline{A}^\times$ such that (1) becomes an isomorphism of groups for every k-scheme T. (If you do this via Yoneda, then give a short argument for how it applies.)

(c) Consider the case $k = \mathbb{R}$ and $A = \mathbb{C}$; set $G = \underline{A}^\times$. Define a group scheme homomorphism $N : G \to \mathbb{G}_{m,\mathbb{R}}$ such that

$$N(\mathbb{R}) : \mathbb{C}^\times \longrightarrow \mathbb{R}^\times$$

is the norm map $z \mapsto z\bar{z}$. Describe the affine scheme $\ker(N)$ by equations.

Problem 2

Let k be a field. Recall that $\mathbb{G}_{a,k} = \operatorname{Spec} k[t]$ with addition law $a^*(t) = t \otimes 1 + 1 \otimes t$.

(a) Assume that $\operatorname{char}(k) = 0$. Show that $k \xrightarrow{\sim} \operatorname{End}(\mathbb{G}_{a,k})$ via

$$\lambda \longmapsto \operatorname{Spec}(t \mapsto \lambda t).$$

(b) Now assume that $\operatorname{char}(k) = p$. Show that $f = \operatorname{Spec} f^*$, where $f^* : k[t] \to k[t]$ is any k-algebra morphism, lies in $\operatorname{End}(\mathbb{G}_{a,k})$ if and only if $f^*(t)$ is of the form

$$f^*(t) = a_n t^{p^n} + a_{n-1} t^{p^{n-1}} + \ldots + a_1 t^p + a_0 t$$

for some $n \geq 0$ and coefficients $a_0, \ldots, a_n \in k$.

Further Problems

Problem 3 (Orthogonal and unitary groups)

(a) Let k be a field with $\mathrm{char}(k) \neq 2$ and let $H \in M_n(k)$ be a symmetric matrix. Construct a k-group scheme $O(H)$ that represents the functor

$$T \longmapsto \left\{ g \in GL_n(\mathcal{O}_T(T)) \mid {}^t g H g = H \right\}.$$

It is called the orthogonal group of H. Prove that the determinant defines a group scheme morphism $O(H) \to \mu_{2,k}$. Its kernel is the special orthogonal group $SO(H)$.

(b) Let K/k be a separable quadratic extension with Galois conjugation σ. Let $H \in M_n(K)$ be a hermitian matrix, meaning $\sigma({}^t H) = H$. Construct a k-group scheme $U(H)$ that represents the functor

$$T \longmapsto \left\{ g \in GL_n(K \otimes_k \mathcal{O}_T(T)) \mid (\sigma \otimes 1)({}^t g) H g = H \right\}.$$

It is called the unitary group of H. Show further that there exists an isomorphism of K-group schemes

$$K \otimes_k U(H) \xrightarrow{\sim} GL_{n,K}.$$

Hint: Use the isomorphism $K \otimes_k K \xrightarrow{\sim} K \times K$.

Problem 4 (Frobenius isogeny)

Let R be an $\mathbb{F}_p$-scheme. Recall that we defined $GL_{n,R}$ as

$$GL_{n,R} = \mathrm{Spec}\left(R[t_{ij}, 1 \leq i,j \leq n;\ \det(t_{ij})^{-1}] \right).$$

Let $F_{/R} : GL_{n,R} \to GL_{n,R}$ be the *relative Frobenius morphism over* R. By definition, it is characterized by

$$F_{/R}^*(t_{ij}) = t_{ij}^p, \qquad F_{/R}^*(a) = a, \quad a \in R.$$

In particular, it is a morphism of R-schemes. Show that $F_{/R}$ is an R-group scheme endomorphism of $GL_{n,R}$. Determine $\ker(F_{/R})$ as scheme, and determine the induced endomorphism of $GL_{n,R}(T)$ for every $T \to \mathrm{Spec}\, R$.