Exercise Sheet 3

Discussed on 28.04.2021

Problem 1. Let k be a field and let $f\colon X \to Y$ be a separable[1] map of connected proper smooth curves over k.

(a) Show that there is a natural exact sequence

$$0 \to f^*\Omega^1_{Y/k} \to \Omega^1_{X/k} \to \Omega^1_{X/Y} \to 0$$

of sheaves on X.

Hint: To get injectivity of the first map, look at the stalk at the generic point.

(b) Deduce that $\Omega^1_{X/Y}$ is zero on a dense open subset of X. Assuming that k has characteristic 0, show that for every closed point $x \in X$ we have

$$\dim_k \Omega^1_{X/Y,x} = (e_x - 1) \cdot [\kappa(x) : k],$$

where e_x is the ramification index of f at x. In particular, f is ramified at only finitely many points.

Hint: Use that Ω^1 commutes with localization. Show that $\Omega^1_{\mathcal{O}_{X,x}/k}$ is free over $\mathcal{O}_{X,x}$ and generated by $d\pi$, where $\pi \in \mathcal{O}_{X,x}$ is any uniformizer.

Problem 2. (a) Let E be an elliptic curve over $\mathbb{C}$. Show that for every $N > 0$, $E[N] := \ker([N]\colon E \to E)$ is isomorphic to $(\mathbb{Z}/N\mathbb{Z})^2$.

(b) A *level N-structure* on E is an isomorphism $\alpha\colon (\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{\sim} E[N]$. A morphism $(E, \alpha) \to (E', \alpha')$ of elliptic curves with level N-structures is a morphism $f\colon E \to E'$ of elliptic curves such that $\alpha' = f \circ \alpha$.

Let $\Gamma(N) \subset \mathrm{GL}_2(\mathbb{Z})$ be the kernel of the projection $\mathrm{GL}_2(\mathbb{Z}) \twoheadrightarrow \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. Show that there is a canonical bijection

$$\Gamma(N) \setminus \mathcal{H}^{\pm} \xleftrightarrow{\sim} \{\text{elliptic curves}/\mathbb{C} \text{ with level } N\text{-structure}\}/\cong$$

(c) Show that for $N \geq 4$, the action of $\Gamma(N)$ on $\mathcal{H}^{\pm}$ is free, i.e. all stabilizers are trivial.

[1] "f separable" means that the associated field extension $\kappa(\eta_X)/\kappa(\eta_Y)$ of function fields is separable.