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Summer term 2020

Exercises, Algebra I (Commutative Algebra) – Week 11

Exercise 55. (Union of associated prime ideals, 3 points)
Assume $\mathfrak{a} = \bigcap \mathfrak{q}_i$ is a minimal primary decomposition in a Noetherian ring A with associated prime ideals $\mathfrak{p}_i := \sqrt{\mathfrak{q}_i}$. Show that then

$$\bigcup \mathfrak{p}_i = \{a \in A \mid (\mathfrak{a} : a) \neq \mathfrak{a}\} = \{a \in A \mid \bar{a} \in A/\mathfrak{a} \text{ is a zero divisor}\}.$$

Exercise 56. (Products of coprime ideals, 2 points)
Assume $\mathfrak{a}_i$, $i = 1, \ldots, n$, are pairwise coprime ideals, i.e. $\mathfrak{a}_i + \mathfrak{a}_j = (1)$ for $i \neq j$. Show that then

$$\prod_{i=1}^{n} \mathfrak{a}_i = \bigcap_{i=1}^{n} \mathfrak{a}_i.$$

(For $n = 2$ the argument has been given in the proof of Proposition 14.15.)

Exercise 57. (Primary decomposition, 4 points)
Let $A = k[x, y, z]$ and $\mathfrak{a} = (xy, x - yz)$.

(i) Show that $V(\mathfrak{a}) = (V(x) \cup V(y)) \cap V(x - yz) = V(x, y) \cup V(x, z)$.

(ii) Deduce that the set of isolated associated prime ideals is $\{(x, y), (x, z)\}$.

(iii) Find a minimal primary decomposition of $\mathfrak{a}$.

Exercise 58. (Example of a primary ideal, 3 points)
Show that $\mathfrak{m} = (2, x) \subset \mathbb{Z}[x]$ is a maximal ideal, that $\mathfrak{q} = (4, x)$ is $\mathfrak{m}$-primary, but $\mathfrak{q} \neq \mathfrak{m}^k$ for all k.

Exercise 59. (Case of radical ideals, 2 points)
Show that an ideal $\mathfrak{a} \subset A$ which admits a primary decomposition and satisfies $\sqrt{\mathfrak{a}} = \mathfrak{a}$ has no embedded (i.e. non-minimal) associated prime ideals.

Exercise 60. (Primary decomposition, 3 points)
Let $A = k[x, y, z]$ and $\mathfrak{a} = \mathfrak{p}_1\mathfrak{p}_2$ with $\mathfrak{p}_1 = (x, y)$, $\mathfrak{p}_2 = (x, z)$. Set $\mathfrak{m} = (x, y, z)$. Show that $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a minimal primary decomposition of $\mathfrak{a}$. Which prime ideals are isolated and which are embedded?

Solutions to be handed in before Monday June 29, 4pm.