Warning: The difficulty of the following exercises is not uniform: some of these just involve checking definitions while others are more substantial. Some of these exercises may also require you to look up definitions which were not covered in class. Unless otherwise indicated, we always follow the conventions and notation from class (e.g. manifolds are smooth, vector spaces are finite dimensional, etc.)

1. Let Q be a manifold and let $\pi : T^*Q \to Q$ be its cotangent bundle.

   (i) Let $\phi : Q \supset U \to \mathbb{R}^n$ be a chart. Spell out why this induces a natural chart $\pi^{-1}(U) \to \mathbb{R}^{2n}$.

   For historical reasons, one usually writes $(q_1, \dots, q_n)$ for the coordinate functions on the base (meaning $q_i := \pi_i \circ \phi$, where $\pi_i : \mathbb{R}^n \to \mathbb{R}$ is the projection onto the i-th component), and $(q, p) = (q_1, \dots, q_n, p_1, \dots, p_n)$ for the induced coordinates on the total space, which are called *canonical coordinates.*

   (ii) The *canonical* 1*-form* $\lambda_{can} \in \Omega^1(T^*Q)$ is defined as follows: given a point $w \in T^*Q$ and a tangent vector $v \in T_w(T^*Q)$, one lets

   $$(\lambda_{can})_w(v) := w(d\pi(v))$$

   Check carefully that you understand the meaning of this expression. The canonical 1-form is often called the *Liouville* 1*-form* in the literature (and probably also in this class. . .)

   (iii) Express λ_{can} in canonical coordinates with respect to some chart on Q, and verify that $d\lambda_{can}$ is a symplectic form.

   (iv) Verify that the zero section and the cotangent fibers are Lagrangian submanifolds (in fact, they are *exact* Lagrangian submanifolds, meaning that the canonical 1-form restricts to an exact 1-form).

   Remark. In Hamiltonian mechanics, classical particles are represented by points of T^*Q. More precisely, a point $(q, p) \in T^*Q$ is a classical particle with position $q \in Q$ and momentum p. In contrast, quantum particles are represented by Lagrangian submanifolds. The zero section in T^*Q is a quantum particle whose momentum is zero and whose position is unknown. The cotangent fiber T^*_qQ is a quantum particle whose position is q and whose momentum is unknown. The fact that Lagrangian submanifolds are necessarily of dimension $\frac{1}{2}dim(T^*Q) > 0$ can be viewed as a manifestation of the uncertainty principle.

2. Let (V, ω) be a symplectic vector space of dimension $2n$. Let $W \subset V$ be a vector subspace.
   (i) if $W \subset V$ is isotropic, then $\dim W \leq n$.
   (ii) if $W \subset V$ is coisotropic, then $\dim W \geq n$.
   (iii) if $W \subset V$ is Lagrangian, then $\dim W = n$.
3. Let (V, ω) be a symplectic vector space and let $W \subset V$ be a vector subspace. Verify that $(W^{\perp})^{\perp} = W$.
4. Verify the following:
   (i) a symplectic manifold is necessarily even dimensional
   (ii) a symplectic manifold is naturally oriented
   (iii) if (M^{2n}, ω) is a closed symplectic manifold then $H^{2i}(M, \mathbb{R}) \neq 0$ for $0 \leq i \leq n$.
5. Given a symplectic manifold (M, ω) and a (time-dependent) Hamiltonian $H = (H_t)_{t \in S^1}, H_t : M \to \mathbb{R}$, recall that we defined the associated Hamiltonian vector field $(X_t)_{t \in S^1}$ as the unique solution to *Hamilton's equation*
$$i_{X_t}\omega = -dH_t.$$
   (i) suppose that $(M, \omega) = (\mathbb{R}^{2n}, \omega_0)$. Expand Hamilton's equation into coordinates and check that one recovers the classical form of Hamilton's equations from classical mechanics.
   (ii) recall that a symplectomorphism of (M, ω) is said to be *Hamiltonian* if it arises as the time-1 map of a (time-dependent) Hamiltonian. Prove that the Hamiltonian symplectomorphisms forms a subgroup of the group of all symplectomorphisms of (M, ω).
6. The purpose of this exercise is to explain why any closed form on a closed Riemannian manifold admits a *canonical* primitive. This statement is useful in many situations. One such situation arose in class: if $X = (X_t)_{t \in [0,1]}$ is a family of Hamiltonian vector fields on a closed symplectic manifold (M, ω), meaning that $i_{X_t}\omega$ is exact, I claimed that one can extract a *smooth* family of functions $H = (H_t)_{t \in [0,1]}$ so that $i_{X_t}\omega = -dH_t$.

   Let (M, g) be a closed n-dimensional Riemannian manifold. Then we can consider an inner-product on the space $\Omega^k(M)$ defined by
$$(\alpha, \beta) := \int_M \langle \alpha, \beta \rangle_g dvol.$$

It can be shown that there is a unique operator $d^* : \Omega^{k+1}(M) \to \Omega^k(M)$ satisfying $(d\alpha, \beta) = (\alpha, d^*\beta)$. A k-form is said to be *harmonic* if it is annihilated by the Hodge Laplacian $dd^* + d^*d : \Omega^k(M) \to \Omega^k(M)$. We let $\mathcal{H}^k(M)$ be the space of harmonic forms.

The *Hodge decomposition theorem* furnishes the following orthogonal decomposition

$$\Omega^k(M) = im(d) \oplus \mathcal{H}^k \oplus im(d^*) \tag{0.1}$$

(i) check that a k-form is harmonic iff it is both closed and co-closed.

(ii) check that d^* restricts to an isomorphism from $im(d) \to im(d^*)$ and similarly d restricts to an isomorphism from $im(d^*) \to im(d)$.

(iii) Conclude that any exact form $\alpha \in im(d)$ admits a canonical primitive (the primitive depends of course on the Riemannian metric).