Simplified Constructibility Theory*

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Preface

This manuscript covers the contents of a minicourse held at the University of Helsinki, Finland in March 2005. We made a fast four-day journey through constructibility theory, starting from first motivations and reaching the heights of morass constructions. This was possible on the basis of *hyperfine structure theory* - a simplified approach to the fine structure theory of the constructible universe. Still, we had to concentrate on the description of fundamental ideas; only few arguments could be carried out completely. This text fills in some more details which are characteristic for the hyperfine theory. It should enable the reader to complete the proofs by himself.

Hyperfine structure theory was introduced by Sy D. friedman and the present author [4]. The standard reference on constructibility theory was written by Keith J. Devlin [3]. Suggestions for further reading on constructibility theory and simplified fine structure are contained in the bibliography.

I am grateful for a mathematically rich and very enjoyable fortnight with the Helsinki logic group, who showed a strong interest in the subject of the course. My special thanks go to Prof. Jouko Väänänen and Prof. Juliette Kennedy for their support and warm hospitality.

Peter Koepke

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1 The continuum problem

The creation of set theory by GEORG CANTOR was initiated and guided by his study of sets of real numbers. In 1873 he showed that the set $\mathbb{R}$ of reals is uncountable [1]. He defined that two sets X, Y have the same cardinality, $\mathrm{card}(X) = \mathrm{card}(Y)$ or $X \sim Y$, if there is a bijection $f: X \leftrightarrow Y$. Thus

$$\mathbb{R} \nsim \mathbb{N}, \text{ i.e., } \neg\exists f\, f: \mathbb{R} \leftrightarrow \mathbb{N}.$$

Further research on the *cardinality* of $\mathbb{R}$ and its subsets lead CANTOR to the formulation of the *continuum hypothesis* (CH) in 1878 [2]:

$$\mathrm{card}(\mathbb{R}) = \aleph_1\,,$$

where $\aleph_0$, $\aleph_1$, ... is the increasing sequence of infinite cardinals. The continuum hypothesis is equivalent to

$$\forall X \subseteq \mathbb{R}\,((\exists n \leqslant \omega\, X \sim n) \vee X \sim \mathbb{R}).$$

In cardinal arithmetic this is usually written as

$$2^{\aleph_0} = \aleph_1\,,$$

where $2^\kappa = \mathrm{card}(\mathcal{P}(\kappa))$ and $\mathbb{R}$ is identified with the powerset of $\mathbb{N}$.

In 1908 FELIX HAUSDORFF [10] extended this to the *generalized continuum hypothesis* (GCH):

$$\forall \alpha\, 2^{\aleph_\alpha} = \aleph_{\alpha+1}\,.$$

Contrary to the expectations of CANTOR, DAVID HILBERT and other mathematicians these hypotheses could not be proved or decided on the basis of the usual set theoretic assumptions and intuitions. To examine *unprovability*, the underlying axioms and logical rules have to be specified; we shall work in ZERMELO-FRAENKEL set theory ZF and usual first-order predicate logic. The system ZF is formulated in the language $\{\in\}$ whose only non-logical symbol is the binary $\in$-relation.

2 GÖDEL's relative consistency results

KURT GÖDEL proved the unprovability of the negation of the continuum hypothesis, i.e., its (relative) consistency, in notes and articles published between 1938 and 1940 [6], [8], [7], [9]. He presents his results in various forms which we can subsume as follows: there is an $\in$-term L such that

$$\mathrm{ZF} \vdash \text{“}(L, \in) \vDash \mathrm{ZF} + \text{the axiom of choice (AC)} + \mathrm{GCH}\text{”}.$$

So ZF sees a model for the stronger theory ZF + AC + GCH. If the system ZF is consistent, then so is ZF+AC+GCH. In ZF, the term L has a host of special properties; L is the $\subseteq$-minimal inner model of ZF, i.e., the $\subseteq$-smallest model of ZF which is transitive and contains the class Ord of ordinals.

The construction of L is motivated by the idea of recursively constructing a minimal model of ZF. The archetypical ZF-axiom is ZERMELO's *comprehension schema* (*axiom of subsets*): for every $\in$-formula $\varphi(v, \vec{w})$ postulate

$$\forall x \forall \vec{p} \ \{v \in x | \varphi(v, \vec{p})\} \in V.$$

The term V denotes the abstraction term $\{v | v = v\}$, i.e., the set theoretic *universe*; formulas with abstraction terms are abbreviations for pure $\in$-formulas. E.g., the above instance of the comprehension schema abbreviates the formula

$$\forall x \forall \vec{p} \ \exists y \forall v (v \in y \leftrightarrow v \in x \wedge \varphi(v, \vec{p})).$$

The basic idea for building a (minimal) model of set theory would be to form a kind of closure under the operations

$$(x, \vec{p}) \longmapsto \{v \in x | \varphi(v, \vec{p})\}.$$

However, there is the difficulty where to evaluate the formula φ. The comprehension instance should be satisfied in the model to be built, i.e., the quantifiers of φ may have to range about sets which have not yet been included in the construction. To avoid this the evaluation of the formula will only refer to sets already constructed and we shall consider the modified definability operations

$$(x, \vec{p}) \longmapsto \{v \in x | (x, \in) \vDash \varphi(v, \vec{p})\}.$$

This could be termed a *predicative* operation whereas the strong operation would be an *impredicative* one. The set $\{v \in x | (x, \in) \vDash \varphi(v, \vec{p})\}$ is determined by fixing x, φ, $\vec{p}$. One can thus view $\{v \in x | (x, \in) \vDash \varphi(v, \vec{p})\}$ as an *interpretation* of the *name* $(x, \varphi, \vec{p})$. These ideas will be used in the definition of the *constructible hierarchy*.

3 The constructible hierarchy

Unlike the impredicative definability operation the predicative operation can be defined as a single set theoretic operation in ZF. We assume that the $\in$-formulas are gödelized by natural numbers in a recursive way so that every formula is $<$-larger than its proper subformulas; let Fml be the set of $\in$-formulas in that Gödelization. We may assume that 0 is the smallest element of Fml. Then define the satisfaction predicate

$$(x, \in) \vDash \varphi(\vec{a})$$

for $(x, \varphi, \vec{a}) \in V \times \mathrm{Fml} \times V$ and the interpretation operation

$$I(x, \varphi, \vec{a}) = \{v \in x | (x, \in) \vDash \varphi(v, \vec{a})\}.$$

One can view

$$\mathrm{Def}(x) = \{I(x, \varphi, \vec{p}) | \varphi \in \mathrm{Fml}, \vec{p} \in x\}$$

as a definable or predicative *powerset* of x. The constructible hierarchy is obtained by iterating the Def-operation along the ordinals.

Definition 1. *Define the* ***constructible hierarchy*** L_α , $\alpha \in \mathrm{Ord}$ *by recursion on* α*:*

$$\begin{aligned} L_0 &= \emptyset \\ L_{\alpha+1} &= \mathrm{Def}(L_\alpha) \\ L_\lambda &= \bigcup_{\alpha<\lambda} L_\alpha \text{, for } \lambda \text{ a limit ordinal.} \end{aligned}$$

The ***constructible universe*** L *is the union of that hierarchy:*

$$L = \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha .$$

The hierarchy satisfies natural hierarchical laws.

Theorem 2. *a)* $\alpha \leqslant \beta$ *implies* $L_\alpha \subseteq L_\beta$

b) L_β *is transitive*

c) $L_\beta \subseteq V_\beta$

d) $\alpha < \beta$ *implies* $L_\alpha \in L_\beta$

e) $L_\beta \cap \mathrm{Ord} = \beta$

f) $\beta \leqslant \omega$ *implies* $L_\beta = V_\beta$

g) $\beta \geqslant \omega$ *implies* $\mathrm{card}(L_\beta) = \mathrm{card}(\beta)$

Proof. By induction on $\beta \in \mathrm{Ord}$. The cases $\beta = 0$ and β a limit ordinal are easy and do not depend on the specific definition of the L_β-hierarchy.

Let $\beta = \gamma + 1$ where the claims hold for γ.

a) It suffices to show that $L_\gamma \subseteq L_\beta$. Let $x \in L_\gamma$. By b), L_γ is transitive and $x \subseteq L_\gamma$. Hence

$$x = \{v \in L_\gamma \mid v \in x\} = \{v \in L_\gamma \mid (L_\gamma, \in) \vDash (v \in w)\frac{x}{w}\} = I(L_\gamma, v \in w, x) \in L_{\gamma+1} = L_\beta .$$

b) Let $x \in L_\beta$. Let $x = I(L_\gamma, \varphi, \vec{p}\,)$. Then by a) $x \subseteq L_\gamma \subseteq L_\beta$.

c) By induction hypothesis,

$$L_\beta = \mathrm{Def}(L_\gamma) \subseteq \mathcal{P}(L_\gamma) \subseteq \mathcal{P}(V_\gamma) = V_{\gamma+1} = V_\beta .$$

d) It suffices to show that $L_\gamma \in L_\beta$.

$$L_\gamma = \{v \in L_\gamma | v = v\} = \{v \in L_\gamma | (L_\gamma, \in) \vDash v = v\} = I(L_\gamma, v = v, \emptyset) \in L_{\gamma+1} = L_\beta .$$

e) $L_\beta \cap \mathrm{Ord} \subseteq V_\beta \cap \mathrm{Ord} = \beta$. For the converse, let $\delta < \beta$. If $\delta < \gamma$ the inductive hypothesis yields that $\delta \in L_\gamma \cap \mathrm{Ord} \subseteq L_\beta \cap \mathrm{Ord}$. Consider the case $\delta = \gamma$. We have to show that $\gamma \in L_\beta$. There is a formula $\varphi(v)$ which is Σ_0 and formalizes being an ordinal. This means that all quantifiers in φ are bounded and if z is transitive then

$$\forall v \in z\, (v \in \mathrm{Ord} \leftrightarrow (z, \in) \vDash \varphi(v)).$$

By induction hypothesis

$$\begin{aligned}\gamma &= \{v \in L_\gamma | v \in \mathrm{Ord}\} \\ &= \{v \in L_\gamma \,|\, (L_\gamma, \in) \vDash \varphi(v)\} \\ &= I(L_\gamma, \varphi, \emptyset) \\ &\in L_{\gamma+1} = L_\beta\,.\end{aligned}$$

f) Let $\beta < \omega$. By c) it suffices to see that $V_\beta \subseteq L_\beta$. Let $x \in V_\beta$. By induction hypothesis, $L_\gamma = V_\gamma$. $x \subseteq V_\gamma = L_\gamma$. Let $x = \{x_0, \ldots, x_{n-1}\}$. Then

$$\begin{aligned}x &= \{v \in L_\gamma \,|\, v = x_0 \vee v = x_1 \vee \ldots \vee v = x_{n-1}\} \\ &= \{v \in L_\gamma \,|\, (L_\gamma, \in) \vDash (v = v_0 \vee v = v_1 \vee \ldots \vee v = v_{n-1}) \frac{x_0\, x_1 \ldots x_{n-1}}{v_0\, v_1 \ldots v_{n-1}}\} \\ &= I(L_\gamma, (v = v_0 \vee v = v_1 \vee \ldots \vee v = v_{n-1}), x_0, x_1, \ldots, x_{n-1}) \\ &\in L_{\gamma+1} = L_\beta\,.\end{aligned}$$

g) Let $\beta > \omega$. By induction hypothesis $\mathrm{card}(L_\gamma) = \mathrm{card}(\gamma)$. Then

$$\begin{aligned}\mathrm{card}(\beta) &\leqslant \mathrm{card}(L_\beta) \\ &\leqslant \mathrm{card}(\{I(L_\gamma, \varphi, \vec{p}) | \varphi \in \mathrm{Fml}, \vec{p} \in L_\gamma\}) \\ &\leqslant \mathrm{card}(\mathrm{Fml}) \cdot \mathrm{card}({}^{<\omega}L_\gamma) \\ &\leqslant \mathrm{card}(\mathrm{Fml}) \cdot \mathrm{card}(L_\gamma)^{<\omega} \\ &= \aleph_0 \cdot \mathrm{card}(\gamma)^{<\omega} \\ &= \aleph_0 \cdot \mathrm{card}(\gamma), \text{ since } \gamma \text{ is infinite,} \\ &= \mathrm{card}(\gamma) \\ &= \mathrm{card}(\beta).\end{aligned}$$

□

The properties of the constructible hierarchy immediately imply the following for the constructible universe.

Theorem 3. *a) L is transitive.*

b) $\mathrm{Ord} \subseteq L$.

Theorem 4. $(L, \in)$ *is a model of* ZF.

Proof. We only demonstrate this for a few of the ZF-axioms.
Pairing Axiom. Let $x, y \in L$. Let $x, y \in L_\alpha$. Then

$$\{x, y\} = \{v \in L_\alpha \,|\, (L_\alpha, \in) \vDash (v = v_1 \vee v = v_2) \frac{x\ \ y}{v_1\, v_2}\} \in L_{\alpha+1}\,.$$

Hence $\{x,\ y\} \in L$. The following sequence of implications show that the closure under pairs indeed entails the pairing axiom in L:

$$\begin{aligned}& \forall x, y \in L\ \{x, y\} \in L \\ \rightarrow\ & \forall x, y \in L\, \exists z \in L\ z = \{x, y\} \\ \rightarrow\ & \forall x, y \in L\, \exists z \in L\, \forall v\, (v \in z \leftrightarrow (v = x \vee v = y)) \\ \rightarrow\ & \forall x, y \in L\, \exists z \in L\, \forall v \in L\, (v \in z \leftrightarrow (v = x \vee v = y)) \\ \rightarrow\ & (L, \in) \vDash \forall x, y\ \exists z\ \forall v\ (v \in z \leftrightarrow (v = x \vee v = y)).\end{aligned}$$

Power Set Axiom. Let $x \in L$. By the power set axiom in V: $\mathcal{P}(x) \cap L \in V$. Let $\mathcal{P}(x) \cap L \subseteq L_\alpha$. Then

$$\mathcal{P}(x) \cap L = \{v \in L | v \subseteq x\} = \{v \in L_\alpha | v \subseteq x\} =$$

$$= \{v \in L_\alpha | (L_\alpha, \in) \vDash \forall w (w \in v \rightarrow w \in x)\} \in L_{\alpha+1}$$

Hence $\mathcal{P}(x) \cap L \in L$. This implies the powerset axiom in L:

$$\begin{aligned}
& \forall x \in L\, \mathcal{P}(x) \cap L \in L \\
\rightarrow\ & \forall x \in L\, \exists z \in L\ z = \mathcal{P}(x) \cap L \\
\rightarrow\ & \forall x \in L\, \exists z \in L\, \forall v (v \in z \leftrightarrow v \subseteq x \wedge v \in L) \\
\rightarrow\ & \forall x \in L\, \exists z \in L\, \forall v \in L (v \in z \leftrightarrow v \subseteq x \wedge v \in L), \text{ since } L \text{ is transitive;} \\
\rightarrow\ & \forall x \in L\, \exists z \in L\, \forall v \in L (v \in z \leftrightarrow v \subseteq x) \\
\rightarrow\ & (L, \in) \vDash \forall x\, \exists z\, \forall v (v \in z \leftrightarrow v \subseteq x).
\end{aligned}$$

Subset Scheme. Let $\varphi(v, \vec{v}\,)$ be an $\in$-formula and $x, \vec{p} \in L$. Let $x, \vec{p} \in L_\alpha$. By the LEVY reflection theorem let $\beta \geqslant \alpha$ such that φ is L-L_β-absolute:

$$\forall v, \vec{v} \in L_\beta\, ((L, \in) \vDash \varphi(v, \vec{v}\,) \text{ iff } (L_\beta, \in) \vDash \varphi(v, \vec{v}\,)).$$

Then

$$\{v \in x | (L, \in) \vDash \varphi(v, \vec{p}\,)\} = \{v \in L_\beta | (L_\beta, \in) \vDash (v \in x \wedge \varphi(v, \vec{p}\,))\} \in L_{\beta+1}.$$

Hence $\{v \in x | (L, \in) \vDash \varphi(v, \vec{p}\,)\} \in L$. □

4 Wellordering L

We shall now prove an *external* choice principle and also an *external* continuum hypothesis for the constructible sets. These will later be internalized through the *axiom of constructibility*. Every constructible set x is of the form

$$x = I(L_\alpha, \varphi, \vec{p}\,);$$

$(L_\alpha, \varphi, \vec{p}\,)$ is a *name* for x.

Definition 5. *Define the class of (**constructible**) **names** or **locations** as*

$$\tilde{L} = \{(L_\alpha, \varphi, \vec{p}\,) | \alpha \in \mathrm{Ord}, \varphi(v, \vec{v}\,) \in \mathrm{Fml}, \vec{p} \in L_\alpha, \mathrm{length}(\vec{p}\,) = \mathrm{length}(\vec{v}\,)\}.$$

This class has a natural stratification

$$\tilde{L}_\alpha = \{(L_\beta, \varphi, \vec{p}\,) \in \tilde{L} | \beta < \alpha\} \textit{ for } \alpha \in \mathrm{Ord}.$$

A location of the form $(L_\alpha, \varphi, \vec{p}\,)$ *is called an* α-***location***.

Definition 6. *Define wellorders* $<_\alpha$ *of* L_α *and* $\tilde{<}_\alpha$ *of* $\tilde{L}_\alpha$ *by recursion on* α.

- $<_0 = \tilde{<}_0 = \emptyset$ *is the vacuous ordering on* $L_0 = \tilde{L}_0 = \emptyset$;

- *if* $<_\alpha$ *is a wellordering of* L_α *then define* $\tilde{<}_{\alpha+1}$ *on* $\tilde{L}_{\alpha+1}$ *by:*
  $(L_\beta, \varphi, \vec{x}) \tilde{<}_{\alpha+1} (L_\gamma, \psi, \vec{y})$ *iff*
  $(\beta < \gamma)$ *or* $(\beta = \gamma \wedge \varphi < \psi)$ *or*
  $(\beta = \gamma \wedge \varphi = \psi \wedge \vec{x}$ *is lexicographically less than* $\vec{y}$ *with respect to* $<_\alpha$ *);*
- *if* $\tilde{<}_{\alpha+1}$ *is a wellordering on* $\tilde{L}_{\alpha+1}$ *then define* $<_{\alpha+1}$ *on* $L_{\alpha+1}$ *by:*
  $y <_{\alpha+1} z$ *iff there is a name for* y *which is* $\tilde{<}_{\alpha+1}$*-smaller then every name for* z*.*
- *for limit* λ*, let* $<_\lambda = \bigcup_{\alpha<\lambda} <_\alpha$ *and* $\tilde{<}_\lambda = \bigcup_{\alpha<\lambda} \tilde{<}_\alpha$*.*

This defines two hierarchies of wellorderings linked by the interpretation function I.

Theorem 7. *a)* $<_\alpha$ *and* $\tilde{<}_\alpha$ *are well-defined*

b) $\tilde{<}_\alpha$ *is a wellordering of* $\tilde{L}_\alpha$

c) $<_\alpha$ *is a wellordering of* L_α

d) $\beta < \alpha$ *implies that* $\tilde{<}_\beta$ *is an initial segment of* $\tilde{<}_\alpha$

e) $\beta < \alpha$ *implies that* $<_\beta$ *is an initial segment of* $<_\alpha$

Proof. By induction on $\alpha \in \mathrm{Ord}$. □

We can thus define wellorders $<_L$ and $\tilde{<}$ of L and $\tilde{L}$ respectively:

$$<_L = \bigcup_{\alpha \in \mathrm{Ord}} <_\alpha \quad \text{and} \quad \tilde{<} = \bigcup_{\alpha \in \mathrm{Ord}} \tilde{<}_\alpha$$

Theorem 8. $<_L$ *is a wellordering of* L*.*

5 An external continuum hypothesis

Theorem 9. $\mathcal{P}(\omega) \cap L \subseteq L_{\aleph_1}$*.*

"**Proof**". Let $m \in \mathcal{P}(\omega) \cap L$. By the downward Löwenheim Skolem theorem let $K \prec L$ be a "sufficiently elementary" substructure such that

$$m \in K \text{ and } \mathrm{card}(K) = \aleph_0 .$$

Let $\pi\colon (K, \in) \cong (K', \in)$ be the Mostowski transitivisation of K defined by

$$\pi(u) = \{\pi(v) | v \in u \wedge v \in K\}.$$

$\pi \restriction \omega = \mathrm{id} \restriction \omega$ and

$$\pi(m) = \{\pi(i) | i \in m \wedge i \in X\} = \{\pi(i) | i \in m\} = \{i | i \in m\} = m.$$

A *condensation argument* will show that there is $\eta \in \mathrm{Ord}$ with

$K' = L_\eta$. $\operatorname{card}(\eta) \leqslant \operatorname{card}(L_\eta) = \operatorname{card}(K) = \aleph_0$ and $\eta < \aleph_1$. Hence

$$m \in K' = L_\eta \subseteq L_{\aleph_1}.$$

6 The axiom of constructibility

The constructible universe L is a model of set theory so that all the above constructions and arguments can be redone within L. In particular one can define a constructible hierarchy $(L_\alpha^L)_{\alpha \in \mathrm{Ord}}$ and a constructible universe L^L *within* $(L, \in)$. By systematically studying the complexity of these definitions and its components one can show that they yield the standard notions defined in the universe V

$$(L_\alpha^L)_{\alpha \in \mathrm{Ord}} = (L_\alpha)_{\alpha \in \mathrm{Ord}} \text{ and } L^L = L.$$

The *axiom of constructibility* is the statement $V = L$, i.e., $\forall x \exists \alpha\, x \in L_\alpha$.

Theorem 10. *$L \vDash V = L$.*

Proof. $V^L = L = L^L$. Hence $(L, \in) \vDash V = L$. □

The axiom of constructibility allows to internalize the wellorderability of L and the continuum hypothesis proved above.

Theorem 11. *a) $V = L$ implies* AC *and* CH.

b) L is a model of AC *and* CH.

Proof. a) Assume $V = L$. By Theorem 8, $<_L$ is a wellordering of L. Hence $<_L$ is a wellordering of V. This implies the axiom of choice. Using Theorem 9

$$\operatorname{card}(\mathbb{R}) = \operatorname{card}(\mathcal{P}(\omega)) = \operatorname{card}(\mathcal{P}(\omega) \cap L) \leqslant \operatorname{card}(L_{\aleph_1}) = \operatorname{card}(\aleph_1) = \aleph_1.$$

b) follows immediately since L is a model of ZF and $V = L$. □

7 Condensation

There are various ways of ensuring the condensation property for the structure K as used in the above argument for the continuum hypothesis. We shall only require closure under some basic operations of constructibility theory, in particular the interpretation operator I. An early predecessor for this approach to condensation and to hyperfine structure theory can be found in GÖDEL's 1939 paper [7]:

> Proof: Define a set K of constructible sets, a set O of ordinals and a set F of Skolem functions by the following postulates I-VII:
>
> I. $M_{\omega_\mu} \subseteq K$ and $m \in K$.

II. If $x \in K$, the order of x belongs to O.

III. If $x \in K$, all constants occuring in the definition of x belong to K.

IV. If $\alpha \in O$ and $\phi_\alpha(x)$ is a propositional function over M_α all of whose constants belong to K, then:

1. The subset of M_α defined by ϕ_α belongs to K.
2. For any $y \in K \cdot M_\alpha$ the designated Skolem functions for ϕ_α and y or $\sim \phi_\alpha$ and y (according as $\phi_\alpha(y)$ or $\sim \phi_\alpha(y)$) belong to F.

V. If $f \in F$, $x_1, ..., x_n \in K$ and $(x_1, ..., x_n)$ belongs to the domain of definition of f, then $f(x_1, ..., x_n) \in K$.

VI. If $x, y \in K$ and $x - y \neq \Lambda$ the first element of $x - y$ belongs to K.

VII. No proper subsets of K, O, F satisfy I--VI.

....

....

Theorem 5. There exists a one-to-one mapping x' of K on M_η such that $x \in y \equiv x' \in y'$ for $x, y \in K$ and $x' = x$ for $x \in M_{\omega_\mu}$.

Proof: The mapping x' (....) is defined by transfinite induction on the order, ...

8 Constructible operations

A substructure of the kind considered by GÖDEL may be obtained as a closure with respect to certain *constructible operations*.

Definition 12. *Define the constructible operations I, N, S by:*

a) ***Interpretation**: for a name $(L_\alpha, \varphi, \vec{x})$ let*
$I(L_\alpha, \varphi, \vec{x}) = \{y \in L_\alpha \mid (L_\alpha, \in) \vDash \varphi(y, \vec{x})\}$;

b) ***Naming**: for $y \in L$ let*
$N(y) =$ the $\tilde{<}$*-least name $(L_\alpha, \varphi, \vec{x})$ such that $I(L_\alpha, \varphi, \vec{x}) = y$.*

c) ***Skolem function**: for a name $(L_\alpha, \varphi, \vec{x})$ let*
$S(L_\alpha, \varphi, \vec{x}) =$ the $<_L$ - least $y \in L_\alpha$ such that $L_\alpha \vDash \varphi(y, \vec{x})$ if such a y exists; set $S(L_\alpha, \varphi, \vec{x}) = 0$ if such a y does not exist.

As we do not assume that α is a limit ordinal and therefore do not have pairing, we make the following convention.

For $X \subseteq L$, $(L_\alpha, \varphi, \vec{x})$ a name we write $(L_\alpha, \varphi, \vec{x}) \in X$ to mean that L_α and each component of $\vec{x}$ is an element of X.

Definition 13. *$X \subseteq L$ is **constructibly closed**, $X \triangleleft L$, iff X is closed under I, N, S:*

$$\begin{aligned} (L_\alpha, \varphi, \vec{x}) \in X &\longrightarrow I(L_\alpha, \varphi, \vec{x}) \in X \text{ and } S(L_\alpha, \varphi, \vec{x}) \in X, \\ y \in X &\longrightarrow N(y) \in X. \end{aligned}$$

*For $X \subseteq L$, $L\{X\} =$ the $\subseteq$-smallest $Y \supseteq X$ such that $Y \triangleleft L$ is called the **constructible hull** of X.*

The constructible hull $L\{X\}$ of X can be obtained by closing X under the functions I, N, S in the obvious way. Hulls of this kind satisfy certain "algebraic" laws which will be stated later in the context of *fine* hulls. Clearly each L_α is constructibly closed.

Theorem 14. (Condensation Theorem) *Let X be constructibly closed and let π: $X \cong M$ be the* MOSTOWSKI *collapse of X onto the transitive set M. Then there is an ordinal α such that $M = L_\alpha$, and π preserves I, N, S and $<_L$:*

$$\pi\colon (X, \in, <_L, I, N, S) \cong (L_\alpha, \in, <_L, I, N, S).$$

Proof. We first show the legitimacy of performing a MOSTOWSKI collapse.
(1) $(X, \in)$ is extensional.
Proof. Let $x, y \in X$, $x \neq y$. Let $N(x) = (L_\alpha, \varphi, \vec{p}) \in X$ and $N(y) = (L_\beta, \psi, \vec{q}) \in X$.
Case 1. $\alpha < \beta$. Then $x \in L_\beta$ and $(L_\beta, \in) \vDash \exists v(v \in x \nleftrightarrow \psi(v, \vec{q}))$. Let

$$z = S(L_\beta, (v \in u \nleftrightarrow \psi(v, \vec{w})), \frac{x\,\vec{q}}{u\,\vec{w}}) \in X$$

Then $z \in x \nleftrightarrow z \in y$. *qed*(1)

We prove the theorem for $X \subseteq L_\gamma$, by induction on γ. There is nothing to show in case $\gamma = 0$. For γ a limit ordinal observe that

$$\pi = \bigcup_{\alpha < \gamma} \pi \upharpoonright (X \cap L_\gamma)$$

where each $\pi \upharpoonright (X \cap L_\gamma)$ is the MOSTOWSKI collapse of the constructibly closed set $X \cap L_\gamma$ which by induction already satisfies the theorem.

So let $\gamma = \beta + 1$, $X \subseteq L_{\beta+1}$, $X \nsubseteq L_\beta$, and the theorem holds for β. Let

$$\pi\colon (X, \in) \cong (\bar{X}, \in)$$

be the MOSTOWSKI collapse of X. $X \cap L_\beta$ is an $\in$-initial segment of X, hence $\pi \upharpoonright X \cap L_\beta$ is the MOSTOWSKI collapse of $X \cap L_\beta$. $X \cap L_\beta$ is constructibly closed and so by the inductive assumption there is some ordinal $\bar{\beta}$ such that

$$\pi \upharpoonright X \cap L_\beta\colon (X \cap L_\beta, \in, <_L, I, N, S) \cong (L_{\bar{\beta}}, \in, <_L, I, N, S).$$

Note that the inverse map π^{-1}: $L_{\bar{\beta}} \rightarrow L_{\beta}$ is elementary since $X \cap L_{\beta}$ is closed under Skolem functions for L_{β}.
(2) $L_{\beta} \in X$.
Proof. Take $x \in X \setminus L_{\beta}$. Let $N(x) = (L_{\gamma}, \varphi, \vec{p})$. Then $L_{\gamma} \in X$ and $L_{\gamma} = L_{\beta}$ since $x \notin L_{\beta}$. *qed*(2)
(3) $\pi(L_{\beta}) = L_{\bar{\beta}}$.
Proof. $\pi(L_{\beta}) = \{\pi(x) | x \in L_{\beta} \wedge x \in X\} = \{\pi(x) | x \in X \cap L_{\beta}\} = L_{\bar{\beta}}$.
(4) $X = \{I(L_{\beta}, \varphi, \vec{p}) | \vec{p} \in X \cap L_{\beta}\}$.
Proof. $\supseteq$ is clear. For the converse let $x \in X$.
Case 1. $x \in L_{\beta}$. Then $x = I(L_{\beta}, v \in v_1, \frac{x}{v_1})$ is of the required form.
Case 2. $x \in L \setminus L_{\beta}$. Let $N(x) = (L_{\beta}, \varphi, \vec{p})$, noting that the first component cannot be smaller than L_{β}. $\vec{p} \in X$ and $x = I(N(x)) = I(L_{\beta}, \varphi, \vec{p})$ is of the required form. *qed*(4)
(5) Let $\vec{x} \in X$. Then $\pi(I(L_{\beta}, \varphi, \vec{x})) = I(L_{\bar{\beta}}, \varphi, \pi(\vec{x}))$.
Proof.

$$\begin{aligned}
\pi(I(L_{\beta}, \varphi, \vec{x})) &= \{\pi(y) | y \in \pi(I(L_{\beta}, \varphi, \vec{x})) \wedge y \in X\} \\
&= \{\pi(y) | (L_{\beta}, \in) \vDash \varphi(y, \vec{x}) \wedge y \in X\} \\
&= \{\pi(y) | (L_{\bar{\beta}}, \in) \vDash \varphi(\pi(y), \pi(\vec{x})) \wedge y \in X\} \\
&= \{z \in L_{\bar{\beta}} | (L_{\bar{\beta}}, \in) \vDash \varphi(z, \pi(\vec{x}))\} \\
&= I(L_{\bar{\beta}}, \varphi, \pi(\vec{x})).
\end{aligned}$$

qed(5)
(6) $\bar{X} = L_{\bar{\beta}+1}$.
Proof. By (4,5),

$$\begin{aligned}
L_{\bar{\beta}+1} &= \{I(L_{\bar{\beta}}, \varphi, \vec{x}) | \vec{x} \in L_{\bar{\beta}}\} \\
&= \{I(L_{\bar{\beta}}, \varphi, \pi(\vec{p})) | \vec{p} \in X \cap L_{\beta}\}, \text{ since } \pi \upharpoonright X \cap L_{\beta}: X \cap L_{\beta} \cong L_{\bar{\beta}}, \\
&= \{\pi(I(L_{\beta}, \varphi, \vec{p})) | \vec{p} \in X \cap L_{\beta}\} \\
&= \pi''\{I(L_{\beta}, \varphi, \vec{p}) | \vec{p} \in X \cap L_{\beta}\} \\
&= \pi''X = \bar{X}.
\end{aligned}$$

qed(6)
(7) Let $y \in X$. Then $\pi(N(y)) = N(\pi(y))$. This means: if $N(y) = (L_{\delta}, \varphi, \vec{x})$ then $N(\pi(y)) = (\pi(L_{\delta}), \varphi, \pi(\vec{x})) = (L_{\pi(\delta)}, \varphi, \pi(\vec{x}))$.
Proof. Let $N(y) = (L_{\delta}, \varphi, \vec{x})$. Then $y = I(L_{\delta}, \varphi, \vec{x})$ and by (5) we have $\pi(y) = I(L_{\pi(\delta)}, \varphi, \pi(\vec{x}))$. Assume for a contradiction that $(L_{\pi(\delta)}, \varphi, \pi(\vec{x})) \neq N(\pi(y))$. Let $N(\pi(y)) = (L_{\eta}, \psi, \vec{y})$. By the minimality of names we have $(L_{\eta}, \psi, \vec{y}) \tilde{<} (L_{\pi(\delta)}, \varphi, \pi(\vec{x}))$. Then $(L_{\pi^{-1}(\eta)}, \psi, \pi^{-1}(\vec{y})) \tilde{<} (L_{\delta}, \varphi, \vec{x})$. By the minimality of $(L_{\delta}, \varphi, \vec{x}) = N(y)$, $I(L_{\pi^{-1}(\eta)}, \psi, \pi^{-1}(\vec{y})) \neq I(L_{\delta}, \varphi, \vec{x}) = y$. Since π is injective and by (5),

$$\begin{aligned}
\pi(y) &\neq \pi(I(L_{\pi^{-1}(\eta)}, \psi, \pi^{-1}(\vec{y}))) \\
&= I(L_{\eta}, \psi, \vec{y}) \\
&= I(N(y)) = y.
\end{aligned}$$

Contradiction. *qed*(7)
(8) Let $x, y \in X$. Then $x <_L y$ iff $\pi(x) <_L \pi(y)$.
Proof. $x <_L y$ iff $N(x) \tilde{<} N(y)$ iff $\pi(N(x)) \tilde{<} \pi(N(y))$ (since inductively π preserves $<_L$ on $X \cap L_\beta$ and $\tilde{<}$ is canonically defined from $<_L$) iff $N(\pi(x)) \tilde{<} N(\pi(y))$ iff $\pi(x) <_L \pi(y)$. *qed*(8)
(9) Let $(L_\delta, \varphi, \vec{x}) \in X$. Then $\pi(S(L_\delta, \varphi, \vec{x})) = S(L_{\pi(\delta)}, \varphi, \pi(\vec{x}))$.
Proof. We distinguish cases according to the definition of $S(L_\delta, \varphi, \vec{x})$.
Case 1. There is no $v \in I(L_\delta, \varphi, \vec{x})$, i.e., $I(L_\delta, \varphi, \vec{x}) = \emptyset$ and $S(L_\delta, \varphi, \vec{x}) = \emptyset$. Then by (5),

$$I(L_{\pi(\delta)}, \varphi, \pi(\vec{x})) = \pi(I(L_\delta, \varphi, \vec{x})) = \pi(\emptyset) = \emptyset$$

and $S(L_{\pi(\delta)}, \varphi, \pi(\vec{x})) = \emptyset$. So the claim holds in this case.
Case 2. There is $v \in I(L_\delta, \varphi, \vec{x})$, and then $S(L_\delta, \varphi, \vec{x})$ is the $<_L$-smallest element of $I(L_\delta, \varphi, \vec{x})$. Let $y = S(L_\delta, \varphi, \vec{x})$. By (5),

$$\pi(y) \in \pi(I(L_\delta, \varphi, \vec{x})) = I(L_{\pi(\delta)}, \varphi, \pi(\vec{x})).$$

So $S(L_{\pi(\delta)}, \varphi, \pi(\vec{x}))$ is well-defined as the $<_L$-minimal element of $I(L_{\pi(\delta)}, \varphi, \pi(\vec{x}))$. Assume for a contradiction that $S(L_{\pi(\delta)}, \varphi, \pi(\vec{x})) \neq \pi(y)$. Let $z = S(L_{\pi(\delta)}, \varphi, \pi(\vec{x})) \in I(L_{\pi(\delta)}, \varphi, \pi(\vec{x}))$. By the minimality of SKOLEM values, $z <_L \pi(y)$. By (8), $\pi^{-1}(z) <_L y$. Since π is $\in$-preserving, $\pi^{-1}(z) \in I(L_\delta, \varphi, \vec{x})$. But this contradicts the $<_L$-minimality of $y = S(L_\delta, \varphi, \vec{x})$ □

9 The generalized continuum hypothesis in L

Theorem 15. $(L, \in) \vDash \mathrm{GCH}$.

Proof. $(L, \in) \vDash V = L$. It suffices to show that

$$\mathrm{ZFC} + V = L \vdash \mathrm{GCH}.$$

Let $\omega_\mu \geqslant \aleph_0$ be an infinite cardinal.
(1) $\mathcal{P}(\omega_\mu) \subseteq L_{\omega_\mu^+}$.
Proof. Let $m \in \mathcal{P}(\omega_\mu)$. Let $K = L\{L_{\omega_\mu} \cup \{m\}\}$ be the constructible hull of $L_{\omega_\mu} \cup \{m\}$. By the Condensation Theorem take an ordinal η and and the MOSTOWSKI isomorphism

$$\pi \colon (K, \in) \cong (L_\eta, \in).$$

Since $L_{\omega_\mu} \subseteq K$ we have $\pi(m) = m$.

$$\eta < \mathrm{card}(\eta)^+ = \mathrm{card}(L_\eta)^+ = \mathrm{card}(K)^+ = \mathrm{card}(L_{\omega_\mu})^+ = \omega_\mu^+.$$

Hence $m \in L_\eta \subseteq L_{\omega_\mu^+}$. *qed*(1)

Thus $\omega_\mu^+ \leqslant \mathrm{card}(\mathcal{P}(\omega_\mu)) \leqslant \mathrm{card}(L_{\omega_\mu^+}) = \omega_\mu^+$. □

10 Fine structure

The hierarchy

$$(L_\alpha, \in, <_L, I, N, S)_{\alpha \in \mathrm{Ord}}$$

of (algebraic) structures satisfies *condensation* in the following sense: any substructure of a level of a level of the hierarchy is isomorphic so some level of the hierarchy. This allows to carry out standard arguments and constructions like GCH, $\diamondsuit$ and KUREPA trees. These results belong to the "coarse" theory of L. RONALD B. JENSEN [11] invented the *fine structure theory* of L, a sophisticated theory in which the exact locations where certain sets are first generated in the constructible universe are studied. An important *fine* principle with many applications in constructions of uncountable structures is JENSEN's *global square principle* $\square$.

Theorem 16. (JENSEN) *Assume $V = L$. Then there exists a system $(C_\beta \mid \beta$ singular$)$ such that*

a) C_β is closed unbounded in β;

b) C_β has ordertype less than β;

c) (coherency) if $\bar{\beta}$ is a limit point of C_β the $\bar{\beta}$ is singular and $C_{\bar{\beta}} = C_\beta \cap \bar{\beta}$.

To prove this one would like to choose a *square sequence* C_β for a given β in a very canonical way, say of minimal complexity or at a minimal place in L. The coherency property is difficult to arrange, it will come out of an involved condensation argument with a structure *in* which β is still regular but *over* which the singularity of β becomes apparent.

Let us consider the process of singularisation of β in L in detail. Let $L \vDash \beta$ is singular. Let γ be minimal such that over L_γ we can define a cofinal subset C of β of smaller ordertype; we can assume that C takes the form

$$C = \{z \in \beta \mid \exists x < \alpha \colon z \text{ is } <_L\text{-minimal such that } L_\gamma \vDash \varphi(z, \vec{p}, x)\}$$

where $\alpha < \beta$, φ is a first order formula, and $\vec{p}$ is a parameter sequence from L_γ. Using the SKOLEM function S we can write this as

$$C = \{S(L_\gamma, \varphi, \vec{p}^\frown x) \mid x < \alpha\}.$$

Here the locations $(L_\gamma, \varphi, \vec{p}^\frown x)$ are $\tilde{<}$-cofinal in the location $(L_\gamma, \varphi, \vec{p}^\frown \alpha)$. The singularization of β may thus be carried out with the SKOLEM function S *restricted to arguments smaller* than $(L_\gamma, \varphi, \vec{p}^\frown \alpha)$. This suggests to say that β is singularised *at the location* $(L_\gamma, \varphi, \vec{p}^\frown \alpha)$ and that the adequate singularizing structure for β is of the form

$$L_{(L_\gamma, \varphi, \vec{p}^\frown \alpha)} = (L_\gamma, \in, <_L, I, N, S \restriction (L_\gamma, \varphi, \vec{p}^\frown \alpha));$$

where $S \restriction (L_\gamma, \varphi, \vec{p}^\frown \alpha)$ means that we have the function $S \restriction L_\gamma$ available as well as the SKOLEM assignments $S(L_\gamma, \psi, \vec{q})$ for $(L_\gamma, \psi, \vec{q}) \tilde{<} (L_\gamma, \varphi, \vec{p}^\frown \alpha)$.

These structures are indexed by locations and provide us with a fine interpolation between successive L_γ-levels:

$$L_\gamma, \ldots, L_{(L_\gamma, \varphi, \vec{p}\,\alpha)}, \ldots, L_{\gamma+1}, \ldots.$$

The interpolated *fine* hierarchy is very slow-growing but satisfies condensation and natural hulling properties which will allow the construction of a $\square$-system. The theory of the fine hierarchy is called *hyperfine structure theory* and was developed by SY FRIEDMAN and the present author [4].

11 The fine hierarchy

Definition 17. *For a location $s = (L_\alpha,\ \varphi,\ \vec{x}\,)$ define the* ***restricted*** **SKOLEM** ***function***

$$S \upharpoonright s = S \upharpoonright \{t \in \tilde{L} \,|\, t \tilde{<} s\}.$$

Define the ***fine level***

$$L_s = (L_\alpha, \in, <_L, I, N, S \upharpoonright s).$$

Then $(L_s)_{s \in \tilde{L}}$ is the ***fine hierarchy****, it is indexed along the wellorder $\tilde{<}$.*

This hierarchy is equipped with algebraic hulling operations. To employ the restricted SKOLEM function $S \upharpoonright s$ at "top locations" $t \tilde{<} s$ of the form $t = (L_\alpha, \varphi, \vec{x}\,)$ we pretend that L_α itself is a constant of the structure $L_s = (L_\alpha, \in, <_L, I, N, S \upharpoonright s)$, i.e. considering L_s and some $Y \subseteq L_\alpha$ we write $(L_\alpha, \varphi, \vec{x}\,) \in Y$ iff $\vec{x} \in Y$.

Definition 18. *Let $s = (L_\alpha, \varphi, \vec{x}\,)$ be a location. $Y \subseteq L_\alpha$ is* ***closed*** *in L_s, $Y \lhd L_s$, if Y is an algebraic substructure of L_s, i.e., Y is closed under I, N, and $S \upharpoonright s$. For $X \subseteq L_\alpha$ let $L_s\{X\}$ be the $\subseteq$-smallest $Y \lhd L_s$ such that $Y \supseteq X$; $L_s\{X\}$ is called the L_s-**hull** of X.*

By our convention, $Y \lhd L_s$ means:

$$\begin{aligned}
L_\beta, \vec{x} \in Y &\longrightarrow I(L_\beta, \varphi, \vec{x}\,) \in Y \text{ and } S(L_\beta, \varphi, \vec{x}\,) \in Y,\\
\vec{x} \in Y \wedge (L_\alpha, \varphi, \vec{x}\,) \tilde{<} s &\longrightarrow S(L_\alpha, \varphi, \vec{x}\,) \in Y\\
y \in Y \wedge N(y) = (L_\beta, \varphi, \vec{x}\,) &\longrightarrow L_\beta, \vec{x} \in Y.
\end{aligned}$$

The fine hierarchy with its associated hull operators again satisfies condensation:

Theorem 19. (*Condensation*) *Let $s = (L_\alpha,\ \varphi,\ \vec{x}\,)$ be a location and suppose that $X \lhd L_s$. Then there is a minimal location $\bar{s}$ so that there is an isomorphism*

$$\pi\colon (X, \in, <_L, I, N, S \upharpoonright s) \cong L_{\bar{s}} = (L_{\bar{\alpha}}, \in, <_L, I, N, S \upharpoonright \bar{s}\,);$$

concerning locations $t \in X$ of the form $t = (L_\alpha, \psi, \vec{y}\,)$ this means

a) $\pi(t) = (L_{\bar{\alpha}}, \psi, \pi(\vec{y}\,))$;

b) $t \tilde{<} s$ *iff* $\pi(t) \tilde{<} \bar{s}$ *and then* $S(\pi(t)) = \pi(S(t))$.

Since π is the MOSTOWSKI *collapse of X the isomorphism π is uniquely determined.*

Proof. Let

$$\pi\colon (X, \in, <_L, I, N, S) \cong (L_{\bar{\alpha}}, \in, <_L, I, N, S)$$

be the unique isomorphism given by the coarse Condensation Theorem 19. Let $\bar{S} = \{\pi(t) | t \in X \wedge t \tilde{<} s\}$.
(1) $\bar{S}$ is an initial segment of $(\tilde{L}, \tilde{<})$.
Proof. Let $\pi(t) \in \bar{S}$, $t \in X$, $t \tilde{<} s$ and $r \tilde{<} \pi(t)$. Let $r = (L_\delta, \psi, \vec{y})$. Since π is surjective there is a location $r' \in X$ such that $r = \pi(r')$. $\pi(r') \tilde{<} \pi(t)$. Since π preserves $<_L$ we have $r' \tilde{<} t \tilde{<} s$. Thus $r \in \bar{S}$. *qed*(1)

Take $\bar{s}$ $\tilde{<}$-minimal such that $\bar{s} \notin \bar{S}$. Then $\bar{S} = \{r \in \tilde{L} | r \tilde{<} \bar{s}\}$. We now have to prove property b) of the theorem. Let $t = (L_\alpha, \psi, \vec{y}) \in X$ be a top location. Then (1) and the definition of $\bar{s}$ imply
(2) $t \tilde{<} s$ iff $\pi(t) \tilde{<} \bar{s}$.

Assume that $t \tilde{<} s$.
(3) $S(\pi(t)) = \pi(S(t))$.
Proof. Let $x = S(t)$, i.e., x is the $<_L$-smallest element of L_α such that

$$(L_\alpha, \in) \vDash \psi(x, \vec{y}).$$

Since $X \lhd L_s$ we have $x \in X$. One can be show by induction on the subformulas of ψ that the map $\pi^{-1}\colon (L_{\bar{\alpha}}, \in) \to (L_\alpha, \in)$ is elementary for every subformula. This is clear for atomic formulas and for propositional connectives; if the subformula is of the form $\exists v \chi$ then $\chi < \psi \leqslant \varphi$ in Fml and X is closed under the Skolem function $S(L_\alpha, \chi, .)$ for the formula $\exists v \chi$; hence π^{-1} is elementary for $\exists v \chi$.

Therefore,

$$(L_{\bar{\alpha}}, \in) \vDash \psi(\pi(x), \pi(\vec{y})),$$

and $S(\pi(t)) = S(L_{\bar{\alpha}}, \psi, \pi(\vec{y}))$ is defined as the $<_L$-minimal $z \in L_{\bar{\alpha}}$ such that

$$(L_{\bar{\alpha}}, \in) \vDash \psi(z, \pi(\vec{y})).$$

Assume for a contradiction that $z = S(\pi(t)) \neq \pi(x)$. By minimality, $z <_L \pi(x)$. Then $\pi^{-1}(z) <_L x$ and again by the elementarity of ψ with respect to π^{-1}:

$$(L_\alpha, \in) \vDash \psi(\pi^{-1}(z), \vec{y}).$$

But this contradicts the minimal definition of $x = S(t)$. □

12 Fine hulls

We prove a couple of further laws about the hulling operation $L_s\{.\}$ which can be seen as *fundamental laws of fine structure theory*. It is conveivable that these laws can be strengthened so that they alone capture the combinatorial content of L and might allow abstract proofs of combinatorial principles. Some of our laws are well-known for any kind of hull by generating functions. A specific and crucial law of hyperfine structure theory is the *finiteness property* (Theorem 24). It corresponds to a similar property in the theory of Silver machines ([13], see also [12]) which was an older attempt to simplify fine structure theory and which is also characterized by hulls and condensations.

Theorem 20. (Monotonicity) *Consider locations* $s = (L_\alpha, \varphi, \vec{x}) \tilde{\leqslant} t = (L_\beta, \psi, \vec{y})$ *and a set* $X \subseteq L_\alpha$.

a) *If* $\alpha = \beta$ *then* $L_s\{X\} \subseteq L_t\{X\}$.

b) *If* $\alpha < \beta$ *then* $L_s\{X\} \subseteq L_t\{X \cup \{\alpha\}\}$.

Proof. a) holds, since all hulling functions of L_s are available in L_t.
b) Note that $L_\alpha \in L_t\{X \cup \{\alpha\}\}$, since $N(\alpha) = (L_\alpha, ., .)$. Then the hulling function of L_s of the form $I(L_\alpha, ., .)$ and $S(L_\alpha, ., .)$ are also available in $L_t\{X \cup \{\alpha\}\}$. □

The next two theorems are obvious for hulls generated with finitary functions.

Theorem 21. (Compactness) *Let* $s = (L_\alpha, \varphi, \vec{x}) \in \tilde{L}$ *and* $X \subseteq L_\alpha$. *Then*

$$L_s\{X\} = \bigcup \{L_s\{X_0\} \mid X_0 \textit{ is a finite subset of } X\}.$$

Theorem 22. (Continuity in the generators) *Let* $s = (L_\alpha, \varphi, \vec{x}) \in \tilde{L}$ *and let* $(X_i)_{i<\lambda}$ *be a* $\subseteq$*-increasing sequence of subsets of* L_α. *Then*

$$L_s\{\bigcup_{i<\lambda} X_i\} = \bigcup_{i<\lambda} L_s\{X_i\}.$$

Since the fine hierarchy grows discontinuously at limit locations (i.e., limits in $\tilde{<}$) of the form $(L_{\alpha+1}, 0, \emptyset)$, where 0 is the smallest element of Fml, we have to distinguish several constellations for the continuity in the locations.

Theorem 23. (Continuity in the locations)

a) *If* $s = (L_\alpha, 0, \emptyset)$ *is a limit location with* α *a limit ordinal and* $X \subseteq L_\alpha$ *then*

$$L_s\{X\} = L\{X\} = \bigcup_{\beta<\alpha} L_{(L_\beta, 0, \emptyset)}\{X \cap L_\beta\}.$$

b) *If* $s = (L_{\alpha+1}, 0, \emptyset)$ *is a limit location and* $X \subseteq L_\alpha$ *then*

$$\begin{aligned} L_s\{X \cup \{\alpha\}\} \cap L_\alpha &= L\{X \cup \{\alpha\}\} \cap L_\alpha \\ &= \bigcup \{L_r\{X\} \mid r \textit{ is an } \alpha\textit{-location}\}. \end{aligned}$$

c) *If* $s = (L_\alpha, \varphi, \vec{x}) \neq (L_\alpha, 0, \emptyset)$ *is a limit location and* $X \subseteq L_\alpha$ *then*

$$L_s\{X\} = \bigcup \{L_r\{X\} \mid r \textit{ is an } \alpha\textit{-location}, r \tilde{<} s\}.$$

Proof. a) is clear from the definitions since the hull operators considered only use the functions I, N, S.
b) The first equality is clear. The other is proved via two inclusions.
($\supseteq$) If z is an element of the right hand side, z is obtained from elements of X by successive applications of I, N, S and $S(L_\alpha, ., .)$. Since $L_\alpha \in L_s\{X \cup \{\alpha\}\}$, z can be obtained from elements of $X \cup \{\alpha\}$ by applications of I, N, S. Hence z is an element of the left hand side.
($\subseteq$) Consider $z \in L\{X \cup \{\alpha\}\} \cap L_\alpha$. There is a finite sequence

$$y_0, y_1, \ldots, y_k = z$$

which "computes" z in $L\{X \cup \{\alpha\}\}$. In this sequence each y_j is an element of $X \cup \{\alpha\}$ or it is obtained from $\{y_i \,|\, i < j\}$ by using I, N, S:

$$y_j = I(L_\beta, \varphi, \vec{y}\,) \quad \text{or} \quad y_j = S(L_\beta, \varphi, \vec{y}\,) \quad \text{or} \quad y_j \text{ is a component of } N(y) \tag{1}$$

for some $L_\beta, \vec{y}\,, y \in \{y_i \,|\, i < j\}$. We show by induction on $j \leqslant k$:

$$\text{if } y_j \in L_\alpha \text{ then } y_j \in U := \bigcup \{L_r\{X\} \,|\, r \text{ is an } \alpha\text{-location}\}.$$

Soe assume the claim for $i < j$ and that $y_j \in L_\alpha$.
Case 1. $y_j \in X \cup \{\alpha\}$. Then the claim is obvious.
Case 2. $y_j = I(L_\beta, \varphi, \vec{y}\,)$ as in property (1) above. If $\beta < \alpha$, then $\beta, \vec{y} \in U$ by induction hypothesis and hence $y_j \in U$.

If $\beta = \alpha$ then $\vec{y} \in U$ by induction hypothesis. Setting

$$\psi(v, \vec{w}\,) = \forall u (u \in v \leftrightarrow \varphi(u, \vec{w}\,))$$

we obtain $y_j = S(L_\alpha, \psi, \frac{\vec{y}}{\vec{w}}) \in U$.
Case 3. $y_j = S(L_\beta, \varphi, \vec{y}\,)$ as in property (1) above. If $\beta < \alpha$, then $\beta, \vec{y} \in U$ by induction hypothesis and hence $y_j \in U$. If $\beta = \alpha$ then $\vec{y} \in U$ by induction hypothesis and $y_j = S(L_\alpha, \varphi, \vec{y}\,) \in U$.
Case 4. y_j is a component of $N(y_i)$ for some $i < j$ as in property (1) above.
Case 4.1. $y_i \in L_\alpha$. Then $y_i \in U$ by induction hypothesis. As U is closed under N, we have $N(y_i) \in U$. So each component of $N(y_i)$ and in particular y_j is an element of U.
Case 4.2. $y_i \in L_{\alpha+1} \setminus L_\alpha$. Since the values of N and S are "smaller" then corresponding arguments, then $y_i = \alpha$ or it is generated by the I-function: $y_i = I(L_\alpha, \psi, \vec{z}\,)$ where $\vec{z} \in \{y_h | h < i\}$, $\vec{z} \in L_\alpha$, and by inductive assumption $\vec{z} \in U$. Since $\alpha = I(L_\alpha$, "v is an ordinal", $\emptyset)$ we may uniformly assume the case $y_i = I(L_\alpha, \psi, \vec{z}\,)$. The name $N(y_i)$ will be of the form $(L_\alpha, \chi, (c_0, \ldots, c_{m-1}))$.

We claim that $c_0 \in U$: if

$$\chi_0(v_0, \vec{w}\,) \equiv \exists v_1 \ldots \exists v_{m-1} \forall u (\chi(u, v_0, v_1, \ldots, v_{m-1}) \leftrightarrow \psi(u, \vec{w}\,))$$

with distinguished variable v_0 then $c_0 = S(L_\alpha, \chi_0, \frac{\vec{z}}{\vec{w}}\vec{z}\,) \in U$.

We then obtain c_1 in U: if

$$\chi_1(v_1, \vec{w}\,) \equiv \exists v_2 \ldots \exists v_{m-1} \forall u (\chi(u, v_0, v_1, \ldots, v_{m-1}) \leftrightarrow \psi(u, \vec{w}\,))$$

with distinguished variable v_1 then $c_1 = S(L_\alpha, \chi_1, \frac{c_0 \frown \vec{z}}{v_0 \frown \vec{w}}) \in U$.

Proceeding in this fashion we get that $y_j \in U$.
c) Note that any element of $L_s\{X\}$ is generated from X by finitely many applications of the functions of L_s and thus only requires finitely many values $S(r)$ with $r \tilde{<} s$. $\square$

Our final hull property is crucial for fine structural considerations. It states that the fine hierarchy grows in a "finitary" way. By incorporating information into finite generators or parameters one can arrange that certain effects can only take place at limit locations which then allows continuous approximations to that situation.

Theorem 24. (Finiteness Property) *Let s be an α-location and let s^+ be its immediate $\tilde{<}$-successor. Then there exists a set $z \in L_\alpha$ such that for any $X \subseteq L_\alpha$:*

$$L_{s^+}\{X\} \subseteq L_s\{X \cup \{z\}\}.$$

Proof. The expansion from L_s to L_{s^+} means to expand the SKOLEM function $S \upharpoonright s$ to $S \upharpoonright s^+ = (S \upharpoonright s) \cup \{(s, S(s))\}$. So $S \upharpoonright s^+$ provides at most one more possible value, namely $S(s)$. Then $z = S(s)$ is as required. □

13 Definition of a gap-1 morass

Combinatorial principles are general statements of infinitary combinatorics which yield construction principles for infinitary, mostly uncountable structures. The continuum hypothesis or the stronger principle $\diamondsuit$ are enumeration principles for subsets of ω or of ω_1 which can be used in recursive constructions. These principles are provable in the model L by non-finestructural methods.

RONALD JENSEN has developed his fine structure theory with a view towards some stronger combinatorial principles. He could prove the full *gap*-1 *two-cardinal transfer property* in L using the combinatorial principle $\square$:

> if a countable first-order theory T has a model $\mathfrak{A} = (A, B, ...)$ with $\mathrm{card}(A) = \mathrm{card}(B)^+ \geqslant \aleph_1$ then for every infinite cardinal κ T has a model $\mathfrak{A}' = (A', B', ...)$ with $\mathrm{card}(A') = \kappa^+$ and $\mathrm{card}(B') = \kappa$.

JENSEN could also prove the *gap*-2 *transfer* by defining and using *gap*-1 *morasses* in a similar way. We shall demonstrate that gap-1 morasses can be naturally constructed in hyperfine structure theory.

A morass is a commutative tree-like system of ordinals and embeddings. Let us first consider a trivial example of a system which may be used in constructions.

$$(\omega \cdot \alpha, <)_{\alpha \leqslant \omega_1}, (\mathrm{id} \upharpoonright \omega \cdot \alpha)_{\alpha \leqslant \beta \leqslant \omega_1}$$

is obviously a directed system whose final structure $(\omega_1, <)$ is determined by the previous structures, all of which are countable. If we have, e.g., a model-theoretic method which recursively constructs additional structure on the countable limit ordinals $(\omega \cdot \alpha, <)$ which is respected by the maps $\mathrm{id} \upharpoonright \omega \cdot \alpha$ then the system "automatically" yields a limit structure on $(\omega_1, <)$. Of course this is just the standard union-of-chains method, always available in ZFC, which is a main tool for many kinds of infinitary constructions.

A (gap-1) morass may be seen as a commutative system of directed systems. In an $(\omega_1, 1)$-morass a top directed system converges to a structure of size ω_2. That system consists of structures of size ω_1 and is itself the limit of a system of directed systems of size ω_0. In applications one has to determine the countable components of this system of systems. If the connecting maps between the countable components commute sufficiently then the morass "automatically" yields a "limit of limits" of size ω_2, whose properties can be steered by appropriate choices of the countable structures.

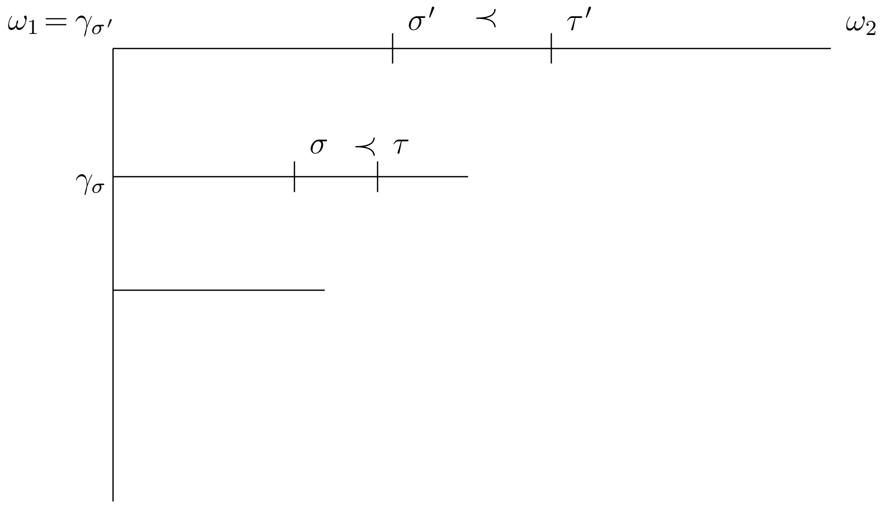
Hyperfine structure theory provides us with a host of structures and structure-preserving maps between them. Through hulls and condensation, one can approximate large structures L_s by countable structures $L_{\bar{s}}$. This motivates the following construction: carefully select a subsystem of the large hyperfine system or category and show that it satisfies JENSEN's structural axioms for an $(\omega_1,\ 1)$-morass. We could instead use arbitrary regular cardinals instead of ω_1. We assume $\mathrm{ZFC} + V = L$ for the rest of this paper.

The following construction is due to the present author and will be published in [5]. We approximate the structure L_{ω_2} by structures which look like L_{ω_2} . The heights of those structure will be *morass points*.

Definition 25. *A limit ordinal $\sigma < \omega_2$ is a **morass point** if*

- $L_\sigma = \bigcup \{L_\mu \mid \mu < \sigma \wedge L_\mu \vDash \mathrm{ZF}^-\}$ *and*
- $L_\sigma \vDash$ *"there is exactly one uncountable cardinal".*

For a morass point σ let γ_σ be the unique uncountable cardinal in L_σ . For morass points σ, τ define $\sigma \prec \tau$ iff $\sigma < \tau$ and $\gamma_\sigma = \gamma_\tau$. Let S^1 be the set of all morass points and $S^0 = \{\gamma_\sigma \mid \sigma \in S^1\}$.



The structures $L_{\sigma'} \subseteq L_{\tau'}$, $\sigma' \prec \tau'$ approximate L_{ω_2} ; the directed system $\sigma' \prec \tau'$ will be a limit of the countable directed systems $\sigma \prec \tau$ from below.

We shall assign levels of the fine hierarchy to morass points; the morass will consist of those levels and of suitable fine maps between them. Finite sets of *parameters* will be important in the sequel and they will often be chosen according to a canonical wellordering "by largest difference":

Definition 26. *Define a wellorder $<^*$ of the class $[V]^{<\omega}$ of finite sets: $p <^* q$ iff there exists $x \in q \setminus p$ such that for all $y >_L x$ holds $y \in p \leftrightarrow y \in q$.*

Lemma 27. *Let σ be a morass point. Then there is a $\tilde{<}$-least location $s(\sigma)$ such that there is a finite set $p \subseteq L_{s(\sigma)}$ with $L_{s(\sigma)}\{\gamma_\sigma \cup p\}$ being cofinal in σ. Let p_σ be the $<^*$-smallest such parameter. We call $M_\sigma = (L_{s(\sigma)}, p_\sigma)$ the **collapsing structure** of σ.*

Proof. Since $L_\sigma \models$ "there is exactly one uncountable cardinal" we have $\sigma \neq \omega_1$. Thus σ is not a cardinal in L. Let $f\colon \gamma_\sigma \to \sigma$ be surjective. Let $f \in L_\alpha$. Set $s = (L_{\alpha+1}, 0, \emptyset) \in \tilde{L}$ and $p = \{f, L_\alpha\}$.
(1) $\sigma \subseteq L_s\{\gamma_\sigma \cup p\}$.
Proof. Let $\zeta \in \sigma$. Let $\zeta = f(\xi)$, $\xi \in \gamma_\sigma$. Then

$$\begin{aligned}\zeta &= \text{the unique set such that } (\xi, \zeta) \in f\\ &= S(L_\alpha, \text{"}(v_1, v_0) \in v_2\text{"}, \frac{\xi\ f}{v_1\, v_2})\\ &\in L_s\{\gamma_\sigma \cup p\}\end{aligned}$$

□

Definition 28. *Define a strict partial order —3 on the set S^1 of morass points: σ—3τ if there exists a structure preserving map*

$$\pi\colon (L_{s(\sigma)}, \in, <_L, I, N, S \upharpoonright s(\sigma)) \to (L_{s(\tau)}, \in, <_L, I, N, S \upharpoonright s(\tau))$$

such that

a) *π is elementary for existential statements of the form $\exists v_0 \ldots \exists v_{m-1}\, \psi$ where ψ is quantifier-free in the language for $(L_{s(\sigma)}, \in, <_L, I, N, S \upharpoonright s(\sigma))$;*

b) *$\pi \upharpoonright \gamma_\sigma = \mathrm{id} \upharpoonright \gamma_\sigma$, $\pi(\gamma_\sigma) = \gamma_\tau > \gamma_\sigma$, $\pi(\sigma) = \tau$, $\pi(p_\sigma) = p_\tau$;*

c) *if τ possesses an immediate $\prec$-predecessor τ' then $\tau' \in \mathrm{range}\, \pi$.*

We shall see that the system (S^1, —3) with connecting maps as in this definition is a gap-1 morass. We first state some results about the "collapsing structures" $(L_{s(\sigma)}, p_\sigma)$.

Lemma 29. *Let $\sigma \in S^1$ be a morass point and $(L_{s(\sigma)}, p_\sigma)$ as defined above. Then*

a) *$s(\sigma)$ is a limit location.*

b) *$\sigma \subseteq L_{s(\sigma)}\{\gamma_\sigma \cup p_\sigma\}$.*

c) *$L_{s(\sigma)}\{\gamma_\sigma \cup p_\sigma\} = L_{s(\sigma)}$.*

Proof. a) Assume for a contradiction that $s(\sigma)$ is a successor location of the form $s(\sigma) = s^+$. By the finiteness property (Theorem 24) there is a $z \in L_s$ such that

$$L_{s(\sigma)}\{\gamma_\sigma \cup p_\sigma\} = L_{s^+}\{\gamma_\sigma \cup p_\sigma\} \subseteq L_s\{\gamma_\sigma \cup p_\sigma \cup \{z\}\}.$$

But then $L_s\{\gamma_\sigma \cup p_\sigma \cup \{z\}\}$ is cofinal in σ, contradicting the minimality of $s(\sigma)$.
b) Let $\xi \in \sigma$. Since $L_\sigma \models$ "γ_σ is the only uncountable cardinal" and $L_{s(\sigma)}\{\gamma_\sigma \cup p_\sigma\}$ is cofinal in σ take $\zeta, L_\eta \in L_{s(\sigma)}\{\gamma_\sigma \cup p_\sigma\}$ such that $\xi < \zeta \in L_\eta$, $\eta < \sigma$, and

$$L_\eta \models \exists f\, f\colon \omega_1 \twoheadrightarrow \zeta \text{ is surjective},$$

where "ω_1" is the ZF-term for the smallest uncountable cardinal. Then

$$g = S(L_\eta, v_0\colon \omega_1 \twoheadrightarrow v_1 \text{ is surjective}, \frac{\zeta}{v_1}) \in L_{s(\sigma)}\{\gamma_\sigma \cup p_\sigma\}$$

is a surjective map $g\colon \gamma_\sigma \twoheadrightarrow \zeta$. Now

$$\xi \in \zeta = \operatorname{range} g \subseteq L_{s(\sigma)}\{\gamma_\sigma \cup p_\sigma\}.$$

c) Let $X = L_{s(\sigma)}\{\gamma_\sigma \cup p_\sigma\} \lhd L_{s(\sigma)}$. By the Condensation Theorem 19 there is a minimal location $\bar{s} \mathrel{\tilde{\preccurlyeq}} s(\sigma)$ so that there is an isomorphism

$$\pi\colon (X, \in, <_L, I, N, S \upharpoonright s(\sigma)) \cong L_{\bar{s}} = (L_{\bar{\alpha}}, \in, <_L, I, N, S \upharpoonright \bar{s}).$$

Since $\sigma \subseteq X$ we have $\pi \upharpoonright \sigma = \mathrm{id} \upharpoonright \sigma$. Let $\bar{p} = \pi(p_\sigma)$. Since π is a homomorphism, $L_{\bar{s}} = L_{\bar{s}}\{\gamma_\sigma \cup \bar{p}\}$. Then $L_{\bar{s}}\{\gamma_\sigma \cup \bar{p}\}$ is trivially cofinal in σ and by the minimal definition of $s(\sigma)$ and p_σ we get $\bar{s} = s(\sigma)$ and $\bar{p} = p_\sigma$. So

$$L_{s(\sigma)} = L_{s(\sigma)}\{\gamma_\sigma \cup p_\sigma\}. \qquad \square$$

Property a) of the preceding Lemma will be crucial; since $s(\sigma)$ is a limit it will be possible to continuous approximate the collapsing structure. The finiteness property of the fine hierarchy makes the hierarchy so slow that most interesting phenomena can be located at limit locations.

Lemma 30. *Let $\sigma \multimap \tau$ witnessed by a structure preserving map*

$$\pi\colon (L_{s(\sigma)}, \in, <_L, I, N, S \upharpoonright s(\sigma)) \to (L_{s(\tau)}, \in, <_L, I, N, S \upharpoonright s(\tau))$$

as in Definition 28. Then π is the unique map satisfying Definition 28.

Proof. Let $x \in L_{s(\sigma)}$. By Lemma 29c, $x = t^{L_{s(\sigma)}}(\vec{\xi}, \vec{p})$ for some $L_{s(\sigma)}$-term t, $\vec{\xi} < \gamma_\sigma$, where $\vec{p}$ is the $<_L$ -increasing enumeration of p_σ . Since π preserves the constructible operations, and since $\pi(\vec{\xi}) = \vec{\xi}$ and $\pi(p_\sigma) = p_\tau$ we have

$$\pi(x) = t^{L_{s(\tau)}}(\vec{\xi}, \vec{q}),$$

where $\vec{q}$ is the $<_L$ -increasing enumeration of p_τ . Hence $\pi(x)$ is uniquely determined by Definition 28. $\square$

This lemma is the basis for

Definition 31. *For $\sigma \multimap \tau$ let*

$$\pi_{\sigma\tau}\colon (L_{s(\sigma)}, \in, <_L, I, N, S \upharpoonright s(\sigma)) \to (L_{s(\tau)}, \in, <_L, I, N, S \upharpoonright s(\tau))$$

be the unique map satisfying Definition 28.

14 Proving the morass axioms

The main theorem states that we have defined a morass in the previous section.

Theorem 32. *The system*

$$(S^1, \sigma \mathrel{-\!\!\!-\!\!\!3} \tau, (\pi_{\sigma\tau})_{\sigma \mathrel{-\!\!\!-\!\!\!3} \tau})$$

is an $(\omega_1, 1)$*-morass, i.e., it satisfies the following axioms:*

– (M0)

  a) *For all* $\gamma \in S^0$ *the set* $S_\gamma = \{\sigma \in S^1 | \gamma_\sigma = \gamma\}$ *is a set of ordinals which is closed in its supremum;*

  b) S_{ω_2} *is closed unbounded in* ω_2 *;*

  c) $S^0 \cap \omega_1$ *is cofinal in* ω_1 *;*

  d) $\mathrel{-\!\!\!-\!\!\!3}$ *is a tree-ordering on* S^1.

– (M1) *Let* $\sigma \mathrel{-\!\!\!-\!\!\!3} \tau$. *Then:*

  a) *Let* $\nu < \sigma$. *Then* ν *is a morass point iff* $\pi_{\sigma\tau}(\nu)$ *is a morass point.*

  b) *For all* $\nu \preccurlyeq \sigma$ *holds:* ν *is* $\prec$*-minimal,* $\prec$*-successor,* $\prec$*-limit iff* $\pi_{\sigma\tau}(\nu)$ *is* $\prec$*-minimal,* $\prec$*-successor,* $\prec$*-limit, respectively.*

  c) *If* τ' *is the immediate* $\prec$*-predecessor of* τ *then* $\pi^{-1}(\tau')$ *is the immediate* $\prec$*-predecessor of* σ.

– (M2) *Let* $\sigma \mathrel{-\!\!\!-\!\!\!3} \tau$, $\bar\sigma \prec \sigma$. *Then* $\bar\sigma \mathrel{-\!\!\!-\!\!\!3} \pi_{\sigma\tau}(\bar\sigma)$ *with correponding map* $\pi_{\bar\sigma\,\pi_{\sigma\tau}(\bar\sigma)} = \pi_{\sigma\tau} \upharpoonright L_{s(\bar\sigma)}$.

– (M3) *Let* $\tau \in S^1$. *Then* $\{\gamma_\sigma \,|\, \sigma \mathrel{-\!\!\!-\!\!\!3} \tau\}$ *is closed in the ordinals* $< \gamma_\tau$.

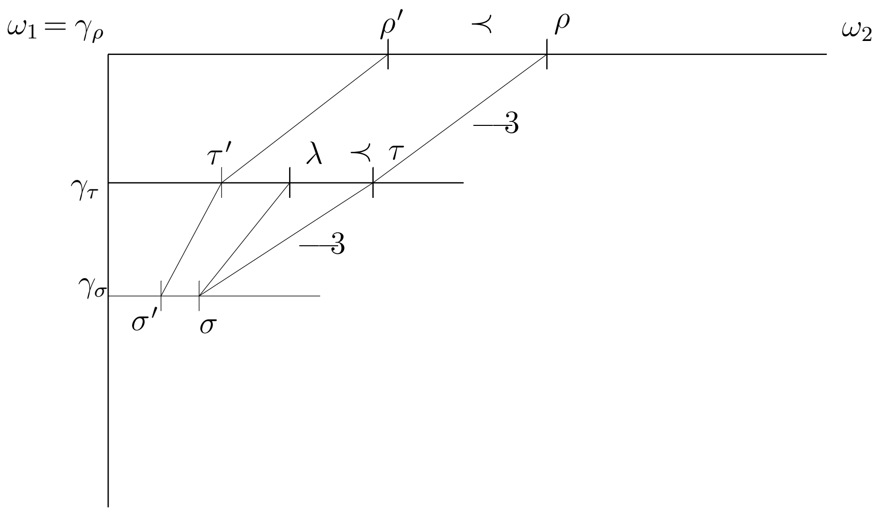
– (M4) *Let* $\tau \in S^1$ *and assume that* τ *is not* $\prec$*-maximal. Then* $\{\gamma_\sigma \,|\, \sigma \mathrel{-\!\!\!-\!\!\!3} \tau\}$ *is cofinal in* γ_τ.

– (M5) *Let* $\{\gamma_\sigma \,|\, \sigma \mathrel{-\!\!\!-\!\!\!3} \tau\}$ *be cofinal in* γ_τ. *Then* $\tau = \bigcup_{\sigma \mathrel{-\!\!\!-\!\!\!3} \tau} \pi_{\sigma\tau}[\sigma]$.

– (M6) *Let* $\sigma \mathrel{-\!\!\!-\!\!\!3} \tau$, σ *a* $\prec$*-limit, and* $\lambda = \sup \operatorname{range} \pi_{\sigma\tau} \upharpoonright \sigma < \tau$. *Then* $\sigma \mathrel{-\!\!\!-\!\!\!3} \lambda$ *with* $\pi_{\sigma\lambda} \upharpoonright \sigma = \pi_{\sigma\tau} \upharpoonright \sigma$.

– (M7) *Let* $\sigma \mathrel{-\!\!\!-\!\!\!3} \tau$, σ *a* $\prec$*-limit, and* $\tau = \sup \operatorname{range} \pi_{\sigma\tau} \upharpoonright \sigma$. *Let* $\alpha \in S^0$ *such that* $\forall \bar\sigma \prec \sigma \exists \bar\upsilon \in S_\alpha\ \bar\sigma \mathrel{-\!\!\!-\!\!\!3} \bar\upsilon \mathrel{-\!\!\!-\!\!\!3} \pi_{\sigma\tau}(\bar\sigma)$. *Then there exists* $\upsilon \in S_\alpha$ *such that* $\sigma \mathrel{-\!\!\!-\!\!\!3} \upsilon \mathrel{-\!\!\!-\!\!\!3} \tau$.

We shall show the morass axioms in a series of lemmas. The axioms can be motivated by the intended applications. Assume that one want to construct a structure of size ω_2 . Take ω_2 as the underlying set of the structure. We present ω_2 as the limit of a system of nicely cohering *countable* structures. The limit process has a two-dimensional nature: *inclusions* $\tau' \prec \tau$ (which implies $\tau' \subseteq \tau$) from left to right and morass maps $\pi_{\sigma\tau}$ going upwards. In the following picture the structure to be put on τ may be considered as enscribed on the vertical axis from 0 to γ_τ and on the horizontal level from γ_τ to τ. In a supposed construction, the horizontal levels are enscribed one after the other from bottom to top. To determine the enscriptions on a level S_α first map up all the enscriptions on levels S_β with $\beta < \alpha$ using the morass maps $\pi_{\sigma\tau}$ with $\sigma \in S_\beta$ and $\tau \in S_\alpha$. Often enough, this does not enscribe all of S_α so that on the non-enscribed parts the structure may be defined according to the specific aim of the construction.



The morass axioms will ensure that the general process above is possible: the morass maps are consistent with each other and with the inclusions from left to right, and the top level will be determined completely from the previous levels. Let us comment on some of the easier axioms. The intention of the complicated axioms M6 and M7 will only become apparent in actual constructions.

- M0 makes some general statements about the morass system: all of ω_2 is covered by morass points; the tree property ensures that a morass point can only be reached by one path from below.

- M1 and M3 give some further information along these lines.

- M2 is necessary for a consistent copying process from lower to higher levels.

- M4 says that a morass point τ which is not maximal on its level is a "limit" of the path leading to it. Together with M5 this completely determines the structure (the enscription) on τ. So the specific construction has to be performed for maximal points σ of levels which are not a limit of the path below.

Lemma 33. (M0) *holds.*

Proof. d) Let $\sigma, \sigma' \mathrel{-\!\!\!-\!\!3} \tau$, $\sigma \leqslant \sigma'$. Then the map $\pi_{\sigma'\tau}^{-1} \circ \pi_{\sigma\tau}\colon L_{s(\sigma)} \to L_{s(\sigma')}$ witnesses that $\sigma \mathrel{-\!\!\!-\!\!3} \sigma'$. So the $\mathrel{-\!\!\!-\!\!3}$-predecessors of any morass point are linearly ordered. Indeed they are wellordered since $\sigma \mathrel{-\!\!\!-\!\!3} \nu$ implies that $\sigma < \nu$. $\square$

Lemma 34. (M1) *holds.*

Proof. (1) Let $\delta \in \mathrm{Ord} \cap L_{s(\sigma)}$. Then $\pi \upharpoonright L_\delta \colon (L_\delta, \in) \to (L_{\pi(\delta)}, \in)$ is elementary.
Proof. For an $\in$-formula φ and $\vec{a} \in L_\delta$ note

$$\begin{aligned}(L_\delta, \in) \models \varphi(\vec{a}) \ \text{ iff } \ & S(L_\delta, \varphi(\vec{w}) \wedge v_0 = 1, \frac{\vec{a}}{\vec{w}}) = 1 \\ \text{iff } \ & S(L_{\pi(\delta)}, \varphi(\vec{w}) \wedge v_0 = 1, \frac{\pi(\vec{a})}{\vec{w}}) = 1 \\ \text{iff } \ & (L_{\pi(\delta)}, \in) \models \varphi(\pi(\vec{a})) \ . \qquad\qquad qed(1)\end{aligned}$$

a) Being a morass point is absolute for transitive ZF$^-$-models. σ is a morass point and so L_σ is a limit of ZF$^-$-models. Take $\delta, \nu < \delta < \sigma$ so that L_δ is a ZF$^-$-model. By (1), $L_{\pi(\delta)}$ is also a ZF$^-$-model. Now ν is a morass point iff $(L_\delta, \in) \models \nu$ is a morass point iff $(L_{\pi(\delta)}, \in) \models \pi(\nu)$ is a morass point iff $\pi(\nu)$ is a morass point.
b) Also being a morass point which is $\prec$-minimal, $\prec$-successor, or $\prec$-limit can be expressed absolutely for ZF$^-$-models and we can use the same technique as in a) to prove preservation.
c) $\pi^{-1}(\tau')$ is defined and it is a morass point by a). Assume for a contradiction that there is a morass point σ', $\pi^{-1}(\tau') \prec \sigma' \prec \sigma$. By a), $\pi(\sigma')$ is a morass point and $\tau' \prec \pi(\sigma') \prec \pi(\sigma) = \tau$, which contradicts the assumptions of c). □

Lemma 35. (M2) *holds*

Proof. Set $\bar{\tau} = \pi_{\sigma\tau}(\bar{\sigma})$. Take $\delta, \bar{\sigma} < \delta < \sigma$ so that L_δ is a ZF$^-$-model. The collapsing structure $(L_{s(\bar{\sigma})}, p(\bar{\sigma}))$ is definable in $(L_\delta, \in)$ from the parameter $\bar{\sigma}$. Then the same terms define $(L_{s(\bar{\tau})}, p(\bar{\tau}))$ in $(L_{\pi_{\sigma\tau}(\delta)}, \in)$ from the parameter $\bar{\tau}$, and the map $\pi_{\sigma\tau}$ restricted to the collapsing structure $L_{s(\bar{\sigma})}$ witnesses $\bar{\sigma} \longrightarrow\!\!\!\!\!\exists\ \pi_{\sigma\tau}(\bar{\sigma})$ by the lementarity of $\pi_{\sigma\tau} \upharpoonright L_\delta \colon (L_\delta, \in) \to (L_{\pi_{\sigma\tau}(\delta)}, \in)$. □

Lemma 36. (M3) *holds*

Proof. Let $\bar{\alpha} < \gamma_\tau$ be a limit of $\{\gamma_\sigma \mid \sigma \longrightarrow\!\!\!\!\!\exists\ \tau\}$. Form the hull

$$L_{s(\tau)}\{\bar{\alpha} \cup \{p_\tau\}\}$$

and by condensation obtain an isomorphism

$$\pi \colon L_{s(\tau)}\{\bar{\alpha} \cup \{p_\tau\}\} \cong L_{\bar{s}} \text{ with } \bar{\tau} = \pi(\tau) \text{ and } \bar{p} = \pi(p_\tau).$$

(1) $\bar{\tau}$ is a morass point.
Proof. Let $\xi < \bar{\tau}$. Take $\sigma \longrightarrow\!\!\!\!\!\exists\ \tau$ such that $\gamma_\sigma < \bar{\alpha}$ and

$$\pi^{-1}(\xi) \in (L_{s(\tau)}\{\gamma_\sigma \cup \{p_\tau\}\} = \mathrm{range}\, \pi_{\sigma\tau} \, .$$

Let $\bar{\xi} < \sigma$ such that $\pi^{-1}(\xi) = \pi_{\sigma\tau}(\bar{\xi})$. Since σ is a morass point take an ordinal μ, $\bar{\xi} < \mu < \sigma$ such that $L_\mu \models \mathrm{ZF}^-$. Then $\pi_{\sigma\tau}(L_\mu) = L_{\pi_{\sigma\tau}(\mu)} \models \mathrm{ZF}^-$.

$$\pi_{\sigma\tau}(L_\mu) = L_{\pi_{\sigma\tau}(\mu)} \in \mathrm{range}\, \pi_{\sigma\tau} \subseteq L_{s(\tau)}\{\bar{\alpha} \cup \{p_\tau\}\} = \mathrm{range}\, \pi^{-1}.$$

Then $\pi(\pi_{\sigma\tau}(L_\mu)) = L_{\pi(\pi_{\sigma\tau}(\mu))}$ is a ZF$^-$-model. Furthermore $\bar{\xi} < \mu < \sigma$ implies that $\pi_{\sigma\tau}(\bar{\xi}) = \pi^{-1}(\xi) < \pi_{\sigma\tau}(\mu) < \pi_{\sigma\tau}(\sigma) = \tau$ and $\xi = \pi(\pi^{-1}(\xi)) < \pi(\pi_{\sigma\tau}(\mu)) < \pi(\tau) = \bar{\tau}$. So $L_{\bar{\tau}}$ is a limit of ZF$^-$-models.

Similarly one can show that $\bar{\alpha}$ is the only uncountable cardinal in $L_{\bar{\tau}}$.

Note that $L_{s(\tau)}\{\bar{\alpha} \cup p_\tau\} \cap \gamma_\tau = \bar{\alpha}$, since $\bar{\alpha}$ is the limit of $L_{s(\tau)}\{\gamma_\sigma \cup p_\tau\} \cap \gamma_\tau = \gamma_\sigma < \bar{\alpha}$. We show $\bar{s} = s(\bar{\tau})$: Clearly $s(\bar{\tau}) \tilde{<} \bar{s}$, since $L_{\bar{s}} = L_{\bar{s}}\{\bar{\alpha} \cup \bar{p}\}$ is cofinal in $\bar{\tau}$. Now assume for a contradiction that $s(\bar{\tau}) \tilde{<} \bar{s}$. Let $\pi_\sigma = \pi \circ \pi_{\sigma\tau}$ for $\sigma \in \{\sigma \mathrel{-\!\!-\!\!3} \tau | \gamma_\sigma < \bar{\alpha}\}$. Choose σ large enough such that there exist $\tilde{s}, \tilde{p} \in L_{s(\sigma)}$ with $s(\bar{\tau}) = \pi_\sigma(\tilde{s})$ and $p_{\bar{\tau}} = \pi_\sigma(\tilde{p})$. By $s(\bar{\tau}) \tilde{<} \bar{s}$ we have $\tilde{s} \tilde{<} s(\sigma)$ and hence $L_{\tilde{s}}\{\gamma_\sigma \cup \tilde{p}\}$ bounded in σ, say by β. But this bound is preserved by $\pi_{\sigma\tau}$ and by π (hence by π_σ); therefore, we get that $L_{s(\bar{\tau})}\{\bar{\alpha} \cup p_{\bar{\tau}}\} \cap \bar{\tau}$ is bounded by $\pi_\sigma(\beta) < \bar{\tau}$ which contradicts the definition of $s(\bar{\tau})$ and $p_{\bar{\tau}}$.

To see that $\pi^{-1}: L_{s(\bar{\tau})} \to L_{s(\tau)}$ is a morass map and hence $\bar{\tau} \mathrel{-\!\!-\!\!3} \tau$ with $\gamma_{\bar{\tau}} = \bar{\alpha}$, we need to show, that π^{-1} preserves Σ_1; the other properties follow by definition, for p_τ and the predecessor of τ (if any) note that $\operatorname{dom} \pi$ contains the ranges of morass maps as subsets.

As a collapsing map, π^{-1} is structure-preserving. Σ_1 is preserved upwards. Now assume, we have a Σ_1-formula in $L_{s(\tau)}$. It is preserved downwards by morass maps $\pi_{\sigma\tau}$ for $\sigma \in \{\sigma \mathrel{-\!\!-\!\!3} \tau | \gamma_\sigma < \bar{\alpha}\}$ and hence has a witness in range $\pi_{\sigma\tau} \subset \operatorname{dom} \pi$. □

Lemma 37. (M4) *holds.*

Proof. Let $\upsilon \in S_{\gamma_\tau}$ with $\tau < \upsilon$. Let $\alpha < \gamma_\tau$ be arbitrary and η between τ and υ s.t. $L_{s(\tau)} \in L_\eta$ and $L_\eta \vDash ZF^-$. Let $X \prec L_\eta$ s.t. $L_{s(\tau)}\{\alpha \cup p_\tau\} \cup \{\tau\} \subset X$ and $\bar{\alpha} := X \cap \gamma_\tau \in \gamma_\tau$. Let $\pi: X \cong L_{\bar{\eta}}$, $\sigma = \pi(\tau)$, and $\bar{p} = \pi(p_\tau)$. So σ is a morass point and $\pi^{-1} \upharpoonright L_{s(\sigma)}: L_{s(\sigma)} \to L_{s(\tau)}$ is elementary and therefore a morass map. Hence $\sigma \mathrel{-\!\!-\!\!3} \tau$ and $\alpha \leq \gamma_\sigma = \bar{\alpha}$. □

Lemma 38. (M5) *holds.*

Proof. Let $\xi \in \tau \in S^1$. We have $L_{s(\tau)} = L_{s(\tau)}\{\gamma_\tau \cup p_\tau\}$ and by cofinality there exists a $\sigma \mathrel{-\!\!-\!\!3} \tau$ with $\xi \in L_{s(\tau)}\{\gamma_\sigma \cup p_\tau\} = \operatorname{range} \pi_{\sigma\tau}$. □

Lemma 39. (M6) *holds.*

Proof. Let $\tilde{s} = \tilde{<}\text{-lub}\,\{\pi_{\sigma\tau}(t) | t \tilde{<} s(\sigma)\}$. We show that $L_{\tilde{s}}\{\gamma_\tau \cup p_\tau\} \cap \tau = \lambda$: First assume $\lambda_0 \in \lambda$; then there is λ_1 with $\lambda_0 < \lambda_1 < \lambda$ and $\lambda_1 = \pi_{\sigma\tau}(\bar{\lambda}_1)$. Then $L_\sigma \vDash \operatorname{card}(\bar{\lambda}_1) \leq \gamma_\sigma$, hence there exists $\bar{f} \in L_\sigma$ s.t. $\bar{f}: \gamma_\sigma \to \bar{\lambda}_1$ is onto, in particular $\bar{f} \in L_{s(\sigma)}\{\gamma_\sigma \cup p_\sigma\}$. As $s(\sigma)$ is a limit location, we have $\bar{f} \in L_t\{\gamma_\sigma \cup p_\sigma\}$ for some $t \tilde{<} s(\sigma)$. Let $f = \pi_{\sigma\tau}(\bar{f}) \in L_{\pi_{\sigma\tau}(t)}\{\gamma_\tau \cup p_\tau\}$, then $f: \gamma_\tau \to \lambda_1$ is onto, so $\lambda_0 \in \operatorname{range} f$, hence $\lambda_0 \in L_{\tilde{s}}\{\gamma_\tau \cup p_\tau\}$. On the other hand assume $\lambda_0 \in L_{\tilde{s}}\{\gamma_\tau \cup p_\tau\} \cap \tau$, then there is a $t \tilde{<} s(\sigma)$ s.t. $\lambda_0 \in L_{\pi_{\sigma\tau}(t)}\{\gamma_\tau \cup p_\tau\}$. But $L_t\{\gamma_\sigma \cup p_\sigma\} \cap \sigma$ is bounded below σ (by β say), since $t \tilde{<} s(\sigma)$, hence also $L_{\pi_{\sigma\tau}(t)}\{\gamma_\tau \cup p_\tau\} \cap \tau$ is bounded below τ, namely by $\pi_{\sigma\tau}(\beta) < \lambda$. So $\lambda_0 \in \lambda$ as required.

Let $\pi: L_{\tilde{s}}\{\gamma_\tau \cup p_\tau\} \cong L_{s_0}$ and $p_0 = \pi(p_\tau)$ (then $\lambda = \pi(\tau)$). Note that $\lambda \in S_{\gamma_\tau}$. We show $L_{s_0}\{\gamma_\tau \cup p_0\} = L_{s(\lambda)}\{\gamma_\tau \cup p_\lambda\}$:

$s_0 = s(\lambda)$: First note that s_0 singularizes λ, so $s(\lambda) \tilde{<} s_0$. Assume for contradiction that s_0 is strictly greater. As $p_\lambda \in L_{s_0}\{\gamma_\tau \cup p_0\}$, we have $p_\lambda \in L_{s_1}\{\gamma_\tau \cup p_0\}$ where $s(\lambda) \tilde{<} s_1 \tilde{<} s_0$ (and where $\alpha(s(\lambda))$ belongs to $L_{s_1}\{\gamma_\tau \cup p_0\}$ in case $\alpha(s(\lambda)) < \alpha(s_0)$; of course we are using the fact that s_0 is a limit location). Since $L_{s(\lambda)}\{\gamma_\tau \cup p_\lambda\} \subset L_{s_1}\{\gamma_\tau \cup p_0\}$, s_1 singularizes λ. By definition of s_0, $\pi^{-1}(s_1) \tilde{<} \tilde{s}$. Further, by definition of $\tilde{s}$, there is a $t \tilde{<} s(\sigma)$ s.t. $\pi^{-1}(s_1) \tilde{<} \pi_{\sigma\tau}(t)$. By minimality of $s(\sigma)$, $L_t\{\gamma_\sigma \cup p_\sigma\} \cap \sigma$ is bounded below σ (by β say). Hence $L_{\pi_{\sigma\tau}(t)}\{\gamma_\tau \cup p_\tau\} \cap \tau$ is bounded below τ (by $\pi_{\sigma\tau}(\beta)$). Since $\pi^{-1}(s_1) \tilde{<} \pi_{\sigma\tau}(t)$, $L_{\pi^{-1}(s_1)}\{\gamma_\tau \cup p_\tau\} \cap \tau$ is bounded below τ (still by $\pi_{\sigma\tau}(\beta)$). Apply π: $L_{s_1}\{\gamma_\tau \cup p_0\} \cap \lambda$ is bounded below λ (by $\pi \circ \pi_{\sigma\tau}(\beta)$), contradiction.

$p_0 = p_\lambda$: $L_{s(\lambda)} = L_{s(\lambda)}\{\gamma_\tau \cup p_0\}$ is cofinal in λ (as above using $s_0 = s(\lambda)$). Therefore, $p_\lambda \leq^* p_0$. Assume for contradiction that p_0 is strictly greater, then using $p_0 \in L_{s(\lambda)} = L_{s(\lambda)}\{\gamma_\tau \cup p_\lambda\}$ and applying π^{-1} we get $\pi^{-1}(p_\lambda) <^* p_\tau \in L_{\tilde{s}}\{\gamma_\tau \cup \pi^{-1}(p_\lambda)\} \subset L_{s(\tau)}\{\gamma_\tau \cup \pi^{-1}(p_\lambda)\}$. Therefore, $L_{s(\tau)} = L_{s(\tau)}\{\gamma_\tau \cup p_\tau\} = L_{s(\tau)}\{\gamma_\tau \cup \pi^{-1}(p_\lambda)\}$ contradicting the minimality of p_τ.

Let $\pi_0 = \pi \circ \pi_{\sigma\tau}$: $L_{s(\sigma)} \to L_{s(\lambda)}$. π_0 is well-defined as range $\pi_{\sigma\tau} = L_{\tilde{s}}\{\gamma_\sigma \cup p_\tau\} \subset \operatorname{dom} \pi$. Further, $\pi_0(\sigma) = \lambda$ and $\pi_0(p_\sigma) = p_\lambda$. Since λ is a $\prec$-limit, property 28c) of the definition of a morass map is vacuous. Finally, π_0 is Σ_1-preserving: First note that π_0 is structure-preserving. Σ_1 formulas are preserved by π_0 upwards, by π upwards (from $L_{s(\lambda)}$ to $L_{\tilde{s}}\{\gamma_\tau \cup p_\tau\}$), and by $\pi_{\sigma\tau}$ downwards, hence by π_0 both ways. Now $\pi_0 = \pi_{\sigma\lambda}$ is a morass map, hence $\sigma \multimap \lambda$ as required. $\square$

Lemma 40. (M7) *holds.*

Proof. We first show that $L_{s(\tau)}\{\alpha \cup p_\tau\} \cap \gamma_\tau = \alpha$, clearly α is a subset of the left side. For the other direction note that since we assume $\tau = \sup \operatorname{range} \pi_{\sigma\tau} \upharpoonright \sigma$, the argument for (M6) shows that $s(\tau) = \tilde{<}$-lub $\{\pi_{\sigma\tau}(t) | t \tilde{<} s(\sigma)\}$. Let $\xi \in L_{s(\tau)}\{\alpha \cup p_\tau\} \cap \gamma_\tau$, then there is $s_0 \tilde{<} s(\sigma)$ s.t. $\xi \in L_{\pi_{\sigma\tau}(s_0)}\{\alpha \cup p_\tau\} \cap \gamma_\tau$. Working downstairs we have that $L_{s_0}\{\gamma_\sigma \cup p_\sigma\}$ does not collapse σ (by minimality of $s(\sigma) \tilde{>} s_0$). Let π_0: $L_{\bar{s}} = L_{\bar{s}}\{\gamma_\sigma \cup \bar{p}\} \cong L_{s_0}\{\gamma_\sigma \cup p_\sigma\}$ where $\bar{p} = \pi_0^{-1}(p_\sigma)$. Then $\sigma' := \pi_0^{-1}(\sigma) < \sigma$. $L_{\bar{s}}$ cannot collapse σ', else there would be a map from γ_σ onto σ' and hence a map from γ_σ onto σ in $L_{s_0}\{\gamma_\sigma \cup p_\sigma\}$. Therefore, $L_{\bar{s}} \models \mathrm{Card}\sigma'$ and $L_\sigma \models \neg\mathrm{Card}\sigma'$, hence $L_{\bar{s}} \in L_\sigma$. Now, σ is a $\prec$-limit, so there is $\bar{\sigma} \prec \sigma$ s.t. $L_{\bar{s}}, \bar{p} \in L_{s(\bar{\sigma})} = L_{s(\bar{\sigma})}\{\gamma_\sigma \cup p_{\bar{\sigma}}\}$.

We shift the isomorphism π_0 to $L_{s(\tau)}$:

$$\text{``}\pi_{\sigma\tau}(\pi_0)\text{''}: L_{\pi_{\sigma\tau}(\bar{s})}\{\gamma_\tau \cup \pi_{\sigma\tau}(\bar{p})\} \cong L_{\pi_{\sigma\tau}(s_0)}\{\gamma_\tau \cup p_\tau\}$$

We started with $\xi \in L_{\pi_{\sigma\tau}(s_0)}\{\alpha \cup p_\tau\} \cap \gamma_\tau$. Now we apply the isomorphism and infer $\xi \in L_{\pi_{\sigma\tau}(\bar{s})}\{\alpha \cup \pi_{\sigma\tau}(\bar{p})\} \cap \gamma_\tau$ (since $\xi < \gamma_\tau$ it is not moved). Further, $L_{\pi_{\sigma\tau}(\bar{s})}\{\alpha \cup \pi_{\sigma\tau}(\bar{p})\} \cap \gamma_\tau \subset L_{s(\pi_{\sigma\tau}(\bar{\sigma}))}\{\alpha \cup p_{\pi_{\sigma\tau}(\bar{\sigma})}\} \cap \gamma_\tau = \alpha$, where the former holds since $\pi_{\sigma\tau}(\bar{p}) \in L_{\pi_{\sigma\tau}(\bar{\sigma})}\{\gamma_\sigma \cup p_{\pi_{\sigma\tau}(\bar{\sigma})}\}$ and $\pi_{\sigma\tau}(\bar{s}) \tilde{<} s(\pi_{\sigma\tau}(\bar{\sigma}))$ and the latter holds by $\bar{\sigma} \multimap \bar{\upsilon} \multimap \pi_{\sigma\tau}(\bar{\sigma})$ for some $\bar{\upsilon} \in S_\alpha$. Hence $\xi \in \alpha$ as desired.

Now we define π: $L_{s(\tau)}\{\alpha \cup p_\tau\} \cong L_{s'}\{\alpha \cup p'\} = L_{s'}$ where $p' := \pi(p_\tau)$, $\upsilon := \pi(\tau)$. By the previous argument we have $\pi^{-1}(\alpha) = \gamma_\tau$. Using the system of morass maps we have $\upsilon \in S_\alpha$.

We have to show $s' = s(\upsilon)$: $L_{s'} = L_{s'}\{\alpha \cup p'\}$ collapses υ, hence $s(\upsilon) \mathrel{\tilde{<}} s'$. Assume for a contradiction that $s(\upsilon) \mathrel{\tilde{<}} s'$. Since $p_\upsilon \in L_{s'}$ we have that there is an s_0 s.t. $s(\upsilon) \mathrel{\tilde{<}} s_0 \mathrel{\tilde{<}} s'$ and $p_\upsilon \in L_{s_0}\{\alpha \cup p'\}$. Since $\pi_{\sigma\tau}$ and π map locations cofinally this is also true for $\pi_0 := \pi \circ \pi_{\sigma\tau}$ (locations $\tilde{<} s(\sigma)$ are mapped to locations $\tilde{<} s'$). Hence without loss of generality, $s_0 = \pi_0(\bar{s}_0)$ where $\bar{s}_0 \mathrel{\tilde{<}} s(\sigma)$. Therefore, $L_{s(\sigma)} \models$ "$L_{\bar{s}_0}\{\gamma_\sigma \cup p_\sigma\}$ is bounded below σ". This is preserved by $\pi_{\sigma\tau}$: $L_{s(\tau)} \models$ "$L_{\pi_{\sigma\tau}(\bar{s}_0)}\{\gamma_\tau \cup p_\tau\}$ is bounded below τ". Finally, this is preserved by π downwards: $L_{s'} \models$ "$L_{s_0}\{\alpha \cup p'\}$ is bounded below υ", contradicting the definition of $s(\upsilon) \mathrel{\tilde{<}} s_0$.

Finally, we have to show that π^{-1} is Σ_1-preserving, then $\pi^{-1} = \pi_{\upsilon\tau}$ and $\pi_{\sigma\upsilon} = \pi_{\upsilon\tau}^{-1} \circ \pi_{\sigma\tau}$. First note that π is structure-preserving.

Σ_1 is preserved upwards by π^{-1} (i.e., from $L_{s(\upsilon)}$ to $L_{s(\tau)}\{\alpha \cup p_\upsilon\}$). For the other direction, assume $L_{s(\tau)} \models \exists x\, \phi(x, \vec{r})$, where ϕ is quantifier-free and $\vec{r} \in \operatorname{dom} \pi = L_{s(\tau)}\{\gamma_\upsilon \cup p_\tau\}$; we have to show $L_{s(\upsilon)} \models \exists x\, \phi(x, \pi(\vec{r}))$. As before, fix $s_0 \mathrel{\tilde{<}} s(\sigma)$ s.t. $\vec{r} \in L_{\pi_{\sigma\tau}(s_0)}\{\gamma_\upsilon \cup p_\tau\}$ and $w \in L_{\pi_{\sigma\tau}(s_0)}\{\gamma_\tau \cup p_\tau\}$ where w is the least witness for $\exists x\, \phi(x, \vec{r})$. Our aim is to show that γ_τ can be replaced by γ_υ in the latter hull.

Let $\pi_1 : L_{s_0}\{\gamma_\sigma \cup p_\sigma\} \cong L_{\bar{s}} = L_{\bar{s}}\{\gamma_\sigma \cup \bar{p}\}$ where $\bar{p} = \pi_1(p_\sigma)$. As above using type preservation, we shift π_1 to the γ_τ-level, let's call the resulting map π_2: $L_{\pi_{\sigma\tau}(s_0)}\{\gamma_\tau \cup p_\tau\} \cong L_{\pi_{\sigma\tau}(\bar{s})}\{\gamma_\tau \cup \pi_{\sigma\tau}(\bar{p})\}$. Then we have $\pi_2(\vec{r}) \in L_{\pi_{\sigma\tau}(\bar{s})}\{\gamma_\upsilon \cup \pi_{\sigma\tau}(\bar{p})\}$ and $\pi_2(w) \in L_{\pi_{\sigma\tau}(\bar{s})}\{\gamma_\tau \cup \pi_{\sigma\tau}(\bar{p})\}$: $L_{\pi_{\sigma\tau}(\bar{s})} \models \phi(\pi_2(w), \pi_2(\vec{r}))$

Further, also as above, there is a $\bar{\sigma} \prec \sigma$ s.t. $L_{\bar{s}} \in L_{\bar{\sigma}}$ with $\bar{\sigma} \mathrel{-\!\!\!-\!\!3} \bar{\upsilon} \mathrel{-\!\!\!-\!\!3} \bar{\tau} := \pi_{\sigma\tau}(\bar{\sigma})$ and $\pi_2(\vec{r})$, $\pi_{\sigma\tau}(\bar{s})$, $\pi_{\sigma\tau}(\bar{p}) \in \operatorname{range} \pi_{\bar{\upsilon}\bar{\tau}}$. Therefore, $\pi_2(w) \in \operatorname{range} \pi_{\bar{\upsilon}\bar{\tau}}$ and hence by $\pi_{\bar{\upsilon}\bar{\tau}}$ being a morass map, we can replace γ_τ by γ_υ in "$\pi_2(w) \in L_{\pi_{\sigma\tau}(\bar{s})}\{\gamma_\tau \cup \pi_{\sigma\tau}(\bar{p})\}$". Applying π_2^{-1} we get $w \in \operatorname{range} \pi_{\upsilon\tau}$. This proves Σ_1-preservation. $\square$

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