The Affine NilTemperley-Lieb Algebra: Basis, Center, Cellular Structure

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The affine nilTemperley-Lieb algebra

Definition

The affine nilTemperley-Lieb algebra of rank N is the unital associative $\mathbb{C}$-algebra $n\widehat{\mathrm{TL}}_N$

- with generators $a_0, \ldots, a_{N-1}$
- and relations (take all indices modulo N)

$$\begin{aligned} a_i^2 &= 0 && \text{for all } 0 \le i \le N-1, \\ a_i a_j &= a_j a_i && \text{for all } |i-j| > 1, \\ a_i a_{i+1} a_i &= a_{i+1} a_i a_{i+1} = 0 && \text{for all } 0 \le i \le N-1. \end{aligned}$$

Write $a(\underline{j}) = a_{j_1} \ldots a_{j_m}$ for a sequence $\underline{j} = (j_1, \ldots, j_m)$ with $j_k \in \{0, \ldots, N-1\}$.

Example for $n\widehat{\mathrm{TL}}_8$: $\quad a(0\ 2\ 7\ 5\ 4\ 0) = a_0 a_2 a_7 a_5 a_4 a_0 = a_0 a_7 a_0 a_2 a_5 a_4 = 0.$

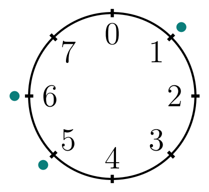
A graphical representation

Let $V = \bigoplus_{n=0}^{N} (\bigwedge^n \mathbb{C}^N) \otimes \mathbb{C}[z]$ with standard wedge basis $v_{j_1} \wedge \ldots \wedge v_{j_n}$ wrt a basis $\{v_1, \ldots, v_N\}$ of $\mathbb{C}^N$.

Action of $n\widehat{\mathrm{TL}}_N$ on V: a_i replaces v_i by v_{i+1}, and 'passing 0' is recorded by multiplication with z.

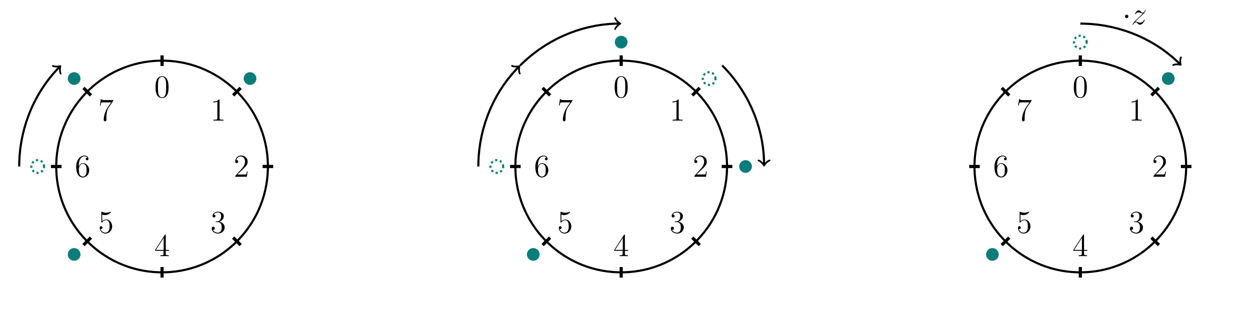
Graphically, V is the $\mathbb{C}[z]$-span of particle configurations with

- up to N particles on a circle with N positions
- at most one particle on each position



$= \quad v_1 \wedge v_5 \wedge v_6$

In the graphical description, a_i moves a particle clockwise from position i to position $i+1$.



(a) $a_6(v_1 \wedge v_5 \wedge v_6) = v_1 \wedge v_5 \wedge v_7$ (b) $a_7 a_1 a_6(v_1 \wedge v_5 \wedge v_6) = v_2 \wedge v_5 \wedge v_0$ (c) $a_0(v_0 \wedge v_5) = z \cdot v_1 \wedge v_5$.

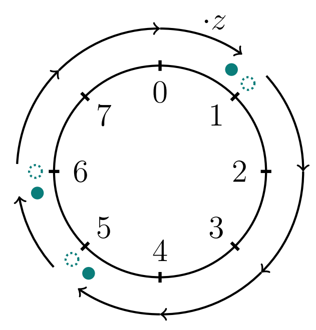
The center

Special elements in $n\widehat{\mathrm{TL}}_N$

For a basis element $v(\underline{k}) \in V$ define a special monomial $a(\widehat{\underline{k}})$ that moves every particle in $v(\underline{k})$ to the position of the preceding particle:

$$a(\widehat{\underline{k}}) v(\underline{k}') = \begin{cases} z \cdot v(\underline{k}) & \text{if } \underline{k}' = \underline{k} \\ 0 & \text{for all } \underline{k}' \neq \underline{k} \text{ (of any length).} \end{cases}$$

$$a(\widehat{1\ 5\ 6}) \cdot v(1\ 5\ 6) = a_4 a_3 a_2 \cdot a_0 a_7 \cdot a_1 a_5 a_6 \ \cdot\ v_1 \wedge v_5 \wedge v_6 = z \cdot v_1 \wedge v_5 \wedge v_6 =$$



For any $1 \le r \le N-1$ define a sum of special monomials $t(r)$ that move r particles placed at any position once around the circle:

$$t(r) := \sum_{|k|=r} a(\widehat{\underline{k}}).$$

Description of the center

Theorem. *The $t(r)$ are central, for all $1 \le r \le N-1$. Moreover,*

- *The center of $n\widehat{\mathrm{TL}}_N$ is generated by 1 and the $t(r)$.*
- *$t(r) \cdot t(m) = 0$ for all $r \neq m$.*

$$\mathrm{Center}(n\widehat{\mathrm{TL}}_N) = \mathbb{C} \oplus t(1) \cdot \mathbb{C}[t(1)] \oplus \ldots \oplus t(N-1) \cdot \mathbb{C}[t(N-1)] \cong \frac{\mathbb{C}[t(1), \ldots, t(N-1)]}{(t(i)t(j) \mid i \neq j)}.$$

The proof relies on FAITHFULNESS of the graphical representation!

Example for $n\widehat{\mathrm{TL}}_3$:

$$\begin{aligned} t(1) &= a_2 a_1 a_0 + a_0 a_2 a_1 + a_1 a_0 a_2, \\ t(2) &= a_0 a_1 a_2 + a_1 a_2 a_0 + a_2 a_0 a_1. \end{aligned}$$

Example for $n\widehat{\mathrm{TL}}_4$:

$$\begin{aligned} t(1) &= a_3 a_2 a_1 a_0 + a_0 a_3 a_2 a_1 + a_1 a_0 a_3 a_2 + a_2 a_1 a_0 a_3, \\ t(2) &= a_0 a_2 a_1 a_3 + a_1 a_3 a_0 a_2 + a_0 a_1 a_3 a_2 + a_1 a_2 a_0 a_3 + a_2 a_3 a_1 a_0 + a_3 a_0 a_2 a_1, \\ t(3) &= a_0 a_1 a_2 a_3 + a_1 a_2 a_3 a_0 + a_2 a_3 a_0 a_1 + a_3 a_0 a_1 a_2. \end{aligned}$$

A basis adapted to the graphical representation

A normal form for monomials in $n\widehat{\mathrm{TL}}_N$

Note that the defining relations of $n\widehat{\mathrm{TL}}_N$ are monomial

$\rightsquigarrow$ enough to search for a basis among monomials – have to find a normal form for monomials!

Note that a monomial $a(\underline{j})$ is nonzero

$\Leftrightarrow$ we have $a(\underline{j}) = \ldots a_i \ldots a_{i\pm1} \ldots a_{i\mp1} \ldots a_i \ldots$ for each two neighbouring a_i in $a(\underline{j})$ (and in between only a_j for $j \neq i-1, i, i+1 \bmod N$).

Algorithm to reorder nonzero monomials	Example for $n\widehat{\mathrm{TL}}_7$: $\quad a(6\ 4\ 2\ 1\ 3\ 5\ 4\ 2\ 0\ 6\ 1\ 3\ 2\ 5)$
Find all a_i without a_{i-1} to their right	$a(6\ 4\ 2\ 1\ 3\ 5\ 4\ 2\ 0\ 6\ \underline{1}\ 3\ \underline{2}\ \underline{5})$
Move them to the very right, don't change their internal order	$a(6\ 4\ 2\ 1\ 3\ 5\ 4\ 2\ 0\ 6\ 3) \cdot a(1\ 2\ 5)$
Repeat	$a(6\ 4\ 2\ 3\ 5\ 4\ 1\ \underline{2}\ 0\ \underline{6}\ \underline{3}) \cdot a(1\ 2\ 5)$ $a(6\ 4\ 2\ 3\ 5\ 4\ 1\ 0) \cdot a(2\ 6\ 3) \cdot a(1\ 2\ 5)$ $a(6\ 4\ 2\ \underline{3}\ 5\ \underline{4}\ 1\ \underline{0}) \cdot a(2\ 6\ 3) \cdot a(1\ 2\ 5)$ $a(6\ 4\ 2\ 5\ 1) \cdot a(3\ 4\ 0) \cdot a(2\ 6\ 3) \cdot a(1\ 2\ 5)$ $a(6\ \underline{4}\ 2\ \underline{5}\ \underline{1}) \cdot a(3\ 4\ 0) \cdot a(2\ 6\ 3) \cdot a(1\ 2\ 5)$ $a(6\ 2) \cdot a(4\ 5\ 1) \cdot a(3\ 4\ 0) \cdot a(2\ 6\ 3) \cdot a(1\ 2\ 5)$
Reorder factors internally according to some global convention	$a(6\ 2) \cdot a(4\ 5\ 1) \cdot a(3\ 4\ 0) \cdot a(2\ 3\ 6) \cdot a(1\ 2\ 5)$

Faithfulness of the graphical representation

Theorem (see [BFZ] for finite case). *For $N \ge 3$, V is a faithful $n\widehat{\mathrm{TL}}_N$-module.*

Sketch of proof:

- Take the basis $\{a(\underline{j}) \mid a(\underline{j}) \text{ in normal form}\}$ of $n\widehat{\mathrm{TL}}_N$ obtained above
- Describe the matrices of $a(\underline{j}) \in \mathrm{End}(V)$ (e.g. block shape!)
- Check their linear independence.

Cellular structure

Affine cellular structure?

An algebra A filtered by two-sided ideals $A = J_n \supset J_{n-1} \supset \ldots \supset J_0 = 0$ coming with an anti-involution i

- is cellular if all subquotients J are cell ideals: $J = \Delta \otimes_{\mathbb{C}} i(\Delta)$ where the left ideal Δ is finite dimensional (+ conditions).
- is affine cellular if all subquotients J are affine cell ideals: $J = \Delta \otimes_{\mathbb{C}} B \otimes_{\mathbb{C}} i(\Delta)$ where the left ideal Δ is finite dimensional and B is a quotient of a polynomial ring (+ conditions).

GOAL: Describe an affine cellular structure on $n\widehat{\mathrm{TL}}_N$!

Full list of simple modules?

Theorem ([KX12]). *For an affine cellular algebra A as above, the simple modules are parametrized by*

$$\{(j, \mathfrak{m}) \mid 1 \le j \le n, \text{ certain maximal ideals } \mathfrak{m} \subset B_j\}.$$

GOAL: Use this theorem to obtain a full list of simple modules of $n\widehat{\mathrm{TL}}_N$!

References

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