GROUPS OF HOMOTOPY SPHERES GRADUATE STUDENT SEMINAR WINTER TERM 2013

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This semester we want to study groups of homotopy spheres. Let θ^n be the group of closed n-manifolds homotopy equivalent to the n-sphere up to h-cobordism. It follows from the h-cobordism theorem that this equals the group of differentiable structures on S^n if $n \geq 5$. The subgroup generated by those homotopy n-spheres which bound parallelizable manifolds will be denoted by bP^{n+1}. Let $J_n \colon \pi_n(SO) \to \pi_n^{st}$ be the stable J-homomorphism, Ω_f^n the group of stably framed closed n-manifolds up to framed cobordism and Σ_f^n the subgroup generated by stably framed homotopy spheres. The goal is to construct for $n \geq 5$ the following exact sequence

$$0 \longrightarrow bP^{n+1} \longrightarrow \theta^n \longrightarrow \mathrm{coker} J_n \longrightarrow \Omega_f^n/\Sigma_f^n \longrightarrow 0. \tag{1}$$

We will see that we can say a lot about the groups bP^n and Ω_f^n/Σ_f^n ($n \geq 5$, $n \neq 6, 14$). For the first group, the question is when the interior of a parallelizable manifold bounding a homotopy sphere can be manipulated in a way such that it becomes contractible while leaving its boundary unchanged. If one can accomplish this, the homotopy sphere one started with is diffeomorphic to the standard sphere by the h-cobordism theorem. Concerning the second group, the question is whether a given stably framed manifold is framed cobordant to a homotopy sphere. It turns out that in either case there are no obstructions in the odd-dimensional case, hence $bP^{2k+1} = \Omega_f^{2k+1}/\Sigma_f^{2k+1} = \{1\}$ ($k > 1$). The obstructions in dimensions $4k$ ($k > 1$) are given by the signature. It follows that bP^{4k} is isomorphic to the subgroup of the integers given by the signatures of parallelizable manifolds which bound homotopy spheres modulo the subgroup generated by the signatures of those parallelizable manifolds with boundary diffeomorphic to the $(4k-1)$-sphere. This number can be calculated in terms of the order of the image of the J-homomorphism in degree $4k-1$ and the kth Bernoulli number. The signature of a stably framed manifold vanishes by the Hirzebruch signature theorem, hence $\Omega_f^{4k}/\Sigma_f^{4k} = \{1\}$. The obstructions in dimensions $4k+2$ ($k \neq 1, 3$) are given by the Kervaire invariant, which implies $bP^{4k+2} \cong \mathbb{Z}/2$ and $\Omega_f^{4k+2}/\Sigma_f^{4k+2} \cong \mathbb{Z}/2$ if there is a closed stably parallelizable $(4k+2)$-manifold with Kervaire invariant one and both groups vanish otherwise.

The image of the J-homomorphism is known due to work of Adams, which leads to computations such as $\theta^7 \cong \mathbb{Z}/28$. Further calculations of θ^n depend on the knowledge

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of the stable homotopy groups of spheres and the solution of the Kervaire invariant one problem.

The main reference will be [2, Chapters VI - X], but it might be helpful to consult the original paper [1] and Levine's lectures [3] as well. A modern treatment using the surgery exact sequence can be found in [4, Chapter 6]. The seminar will be divided into 5 parts.

The first part starts with basic constructions with manifolds, such as attaching handles. We will also discuss plumbing. In the second part we will prove the Handle Presentation Theorem, which asserts that every manifold can be constructed by successive attachment of handles. Based on this theorem we will then prove the h-cobordism theorem in the third part. An immediate consequence is that the group θ^n can be identified with the group of differential structures on S^n if $n \geq 5$. The fourth part will consist of a survey of framed bordism. Finally, in the last part we will discuss surgery, construct the exact sequence (1) and discuss some computations.

TALKS

Talk 1. [17.10. – Dominik Ostermayr] Operations on Manifolds I (p. 89 - 102)
The main goal of this talk is to explain how certain constructions on manifolds can be carried out smoothly. The first example you should discuss is the connected sum of two manifolds (Theorem 1.1). An important consequence of Propositions 1.2 and 1.3 is Corollary 1.4. Together with the h-cobordism theorem it says that homotopy spheres are invertible in the monoid of n-manifolds under connected sum. Time depending, Section 2 may be skipped. More importantly, you should introduce the other constructions such as the boundary connected sum (Section 3), joining two manifolds along submanifolds (Section 4) and joining manifolds along submanifolds along the boundary (Section 5). Proposition 5.3 will be of techichal importance in the course of the proof of the cancellation lemma in Talk 2.

Talk 2. [17.10. – ? ?] Operations on Manifolds II (p. 103 - 110)
A special case of the construction from the first talk is attaching handles, i.e. joining a manifold and a disk along a sphere. This is covered in Section 6. The second part of the talk should be devoted to the proof of the cancellation lemma (Theorem 7.4), which states that sometimes different sequences of attaching handles yield the same result.

Talk 3. [31.10. – Malte Pieper] Operations on Manifolds III (p. 110 - 114)
In the first part you should explain the combinatorial attachment of handles (Section 8), which yields an easier description of the homeomorphism type of a manifold with a handle attached. Then introduce the notion of surgery. Attaching a handle along a sphere to $M \times \{1\} \subset M \times I$ yields the *trace of the surgery*, the upper boundary of which is called the effect of "surgery" (Section 9). Finally, present the results about the homology and the intersection numbers of submanifolds of the trace of a surgery 10.1 -10.5.

Talk 4. [31.10. – Markus Hausmann] Operations on Manifolds IV (p. 115 - 124)
The main subject of this talk is to discuss a certain class of examples of manifolds called (m, k)-handlebodies (of genus g). Those are manifolds which are obtained by attaching g k-handles to the disk D^m. It turns out that if $m > 2k$, those are always the boundary connected sum of $(m-k)$-disc bundles over S^k (Section 11, Proposition 11.2). This leads to a complete classification of $(m, 1)$-handlebodies if $m > 2$ (Corollary 11.4). Time depending, Proposition 11.5 and 11.6 can be skipped. The situation $m = 2k$ is different and is discussed in Section 12. The boundary of a $(2k, k)$-handlebody B turns out to be a homotopy sphere if the *intersection matrix* is unimodular and important homotopy spheres like the *Kervaire spheres* arise this way. The construction of those is usually referred to as *plumbing*.

Talk 5. [14.11. – Alexander Koerschgen] Handle Presentation Theorem I (p. 127 - 130)
The subject of the first talk is to prove Proposition 2.2, which states that given two regular values $a < b$ of a smooth function $f\colon M \to \mathbb{R}$ on a compact manifold, then the cobordism $(f^{-1}(a), f^{-1}[a, b], f^{-1}(b))$ is diffeomorphic to an *elementary cobordism*, i.e. the trace of a surgery, provided f has exactly one critical point in $f^{-1}[a, b]$ and $f^{-1}[a, b] \cap \partial M = \varnothing$.

Talk 6. [14.11. – Christian Wimmer] Handle Presentation Theorem II (p. 126-127, p. 131 - 134)
Start by proving the Handle Presentation Theorem, which says that any cobordism can be obtained by successivley attaching handles (Theorem 1.1 and Corollary 1.2). This results in a *presentation* of the cobordism. By construction one can also obtain the *dual presentation* which geometrically means "turning the cobordism upside down"t. This will be used to prove a Poincaré duality theorem for cobordisms in Talk 7. Construct the chain complex from a presentation of a cobordism which calculates its homology (Theorem 3.4) and make the relation of the chain complex of a presentation with the chain complex of the dual presentation precise (Proposition 3.5).

Talk 7. [28.11. – ? ?] Handle Presentation Theorem III (p. 135 - 138)
In this talk some applications of the Handle Presentation Theorem will be presented. The first one is a theorem of Morse (Theorem 4.1). It says that given a Morse function f on a compact, closed manifold M, we have for any n

$$b_n - b_{n-1} + b_{n-2} - \cdots \leq c_n - c_{n-1} + c_{n-2} - \cdots ,$$

where b_i are the Betti numbers and c_i is the number of critical points of index i. This implies immediately that $\chi(M) = \sum_i (-1)^i c_i$. Another one is the Poincaré duality theorem for cobordisms (Theorem 5.1), i.e. given a cobordism (V_0, W, V_1) such that W is orientable, then $H_i(W, V_1) \cong H^{m-i}(W, V_0)$, where $m = \dim W$. Section 6 contains a technical theorem which ensures that one can always find a presentation with either one or no 0-handle (Theorem 6.1). This has several corollaries for the homotopy groups of a cobordism (6.2 - 6.3) and will be important later, e.g. in the proof of VIII Theorem 1.6.

Talk 8. [28.11. – ? ?] The h-Cobordism Theorem I (p. 143 - 150)
Given a presentation of a cobordism, the differentials are given by matrices with integer coefficients. The main theorem of Section 1 is that under mild assumptions these matrices can be assumed to be lower triangular (Theorem 1.6). Section 2 is concerned with another simplicification, namely removing a row and column which intersect in ± 1 and have zero entries elsewhere. It turns out that this is possible for m-handles where $m \geq 4$ and for 3-handles under a simply connectivity assumption (Theorem 2.3).

Talk 9. [12.12. – Alexander Koerschgen] The h-Cobordism Theorem II (p. 151 - 155)
The problem with cancelling 1- and 2-handles that occured in the previous talk are dealt with in Proposition 3.1. Namely, it is not always possible to cancel such handles, however one can replace them by 3-handles. Now, we will be able to prove the main theorem of this section Theorem 4.1, which asserts that for any cobordism (V_0, W, V_1) such that $H_*(W, V_i)$ are free, there is a presentation such that the number of i-handles equals the ith Betti number $b_i(W, V_0)$. An immediate Corollary is the h-cobordism theorem (Theorem 4.3). This has a couple of Corollaries such as the uniqueness of the smooth structure on the disc D^m ($m \geq 6$) and the Poincaré conjecture for smooth manifolds of dimension larger than 4 (Corollaries 4.5, 4.6). If time permits, you may present 4.7, 4.8 or 4.9.

Talk 10. [12.12. – Ruth Joachimi] The h-Cobordism Theorem III (p. 156 - 160)
Using the h-cobordism theorem, start by showing that m-manifolds modulo h-cobordism form a commutative monoid under the operation of connected sum. The identity is represented by manifolds bounding a contractible manifold and the group of invertible elements is given by the group of homotopy spheres θ^m (Theorem 5.5). Another consequence of the h-cobordism theorem is that for $m \geq 5$, the group of homotopy spheres θ^m is isomorphic to to the group of diffeomorphisms S^{m-1} modulo those which extend over D^m and to the group of invertible differentiable structures on a topological m-sphere. If time permits, you can prove Theorem 6.2. It is a generalization of the Heegaard decomposition of 3-manifolds.

Talk 11. [09.01. – Christian Wimmer] Framed Manifolds I (p. 167 - 182)
The goal of this talk is to give the first part of a survey on framed manifolds. Following Sections 1 - 5, start with a review about framings of vector bundles, stable triviality and parallelizability (Section 1). Then recall briefly the notions of framed submanifolds of a manifold, framed cobordism and some properties (Section 2). Afterwards, introduce the bordism groups $\Omega_f^k(M)$ and compute $\Omega_f^0(M)$ if time permits. Proceed by discussing the Pontriagin construction (Section 5). If time permits you can compute $[M, S^n]$ for n-manifolds M.

Talk 12. [09.01. – Markus Land] Framed Manifolds II (p. 183 - 191)
This talk is the second part of the survey on framed manifolds. To begin with discuss the geometric suspension and state Freudenthal's theorem for the homotopy groups of spheres (6.2). Go on by defining the J-homomorphism, explain how it gives rise to the stable J-homomorphism, give the description of the image of J (6.3.1) as well as

the kernel of the J-homomorphism (6.3.4). Then you should discuss some properties of stable parallelizability (Theorem 7.2). As a consequence of Proposition 7.4, you should show that the Kervaire manifolds are parallelizable (7.5). In the final part you should first introduce *almost* parallelizable manifolds. Show that this is equivalent to stable parallelizability in dimensions not divisible by 4 whereas in dimensions divisible by 4 stable parallelizability is equivalent to the signature to vanish (Theorem 8.5). A corollary is that homotopy spheres are stably parallelizable (Corollary 8.6). At the end you should compute the subgroup of the integers generated by the signatures of almost parallelizable closed $4k$-manifolds (Theorem 8.7).

Talk 13. [23.01. – Malte Pieper] Surgery I (p. 195 - 202)
The aim of the next three talks is to compute the groups bP^{n+1} in the last talk. Given a manifold bounding a homotopy sphere, the aim is to simplify the interior without affecting the boundary. This is done by the method of surgery. In this talk you should first discuss the effect of surgery on homology (Section 1). The main result of Section 2 is that framed surgery below the middle dimension is always possible, i.e. any stably framed manifold of dimension $m \geq 2k > 4$ is framed cobordant to a $(k-1)$-connected manifold.

Talk 14. [23.01. – Markus Hausmann] Surgery II (p. 202 - 209)
This talk is concerened with surgery in the middle dimension of even dimensional manifolds. The question is when a stably framed $2k$-manifold, closed or bounded by a homotopy sphere, is framed cobordant to a k-connected stably framed manifold. If $\dim M = 4n$ $(n > 1)$, the only obstruction is the signature (Theorem 3.4). If $\dim M = 4n+2$ $(n > 1)$, the only obstruction is the Kervaire invariant (Theorem 4.6).

Talk 15. [06.02. – Dominik Ostermayr] Surgery III (p. 210 - 215)
The topic of this talk is surgery on odd-dimensional manifolds. The main result is that any closed stably framed manifold is framed cobordant to a homotopy sphere and if a homotopy sphere Σ is the boundary of a stably framed manifold, then Σ bounds a contractible manifold and is thus diffeomorphic to the sphere by the h-cobordism theorem (Corollaries 5.2, 5.3).

Talk 16. [06.02. – Markus Land] Surgery IV (p. 215 - 219)
In this talk all the pieces are put together to compute bP^{n+1} (Proposition 6.2). By definition it is a subgroup of θ^n and non-triviality implies for instance the existence of exotic smooth structures on spheres. After that explain the sequence 6.1 and how it yields (1). Then you should prove Theorem 6.7 and explain the connection to the Kervaire invariant one problem. Finally, using Adams' calculation of the image of the J-homomorphism, you should calculate θ^n for some values of n.

References

[1] Kervaire, Michel A.; Milnor, John W., *Groups of homotopy spheres. I.*, Ann. of Math. (2) 77 1963 504-537.
[2] Kosinski, Antoni A., *Differential Manifolds*, Dover Publications, 1993.

[3] Levine, J. P., *Lectures on groups of homotopy spheres*, Algebraic and geometric topology (New Brunswick, N.J., 1983), 62?95, Lecture Notes in Math., 1126, Springer, Berlin, 1985.
[4] Lück, Wolfgang, *A basic introduction to surgery theory*, ICTP Lecture Notes Series 9, Band 1, of the school "High-dimensional manifold theory" in Trieste, May/June 2001, Abdus Salam International Centre for Theoretical Physics, Trieste.