UNSTABLE HOMOTOPY THEORY
GRADUATE STUDENT SEMINAR
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The objective of the seminar is to discuss some structural and computational results from unstable homotopy theory. The prerequisites are basic homological algebra and algebraic topology (such as e.g. [6] and [1], respectively). The topic should be interesting for the students working in homotopy theory as well as for people with more geometric background.

The seminar is divided into three main parts. In the first part we discuss the general theory of localizations and completions. As an application we will see how Miller's theorem [2] (which will be stated without proof) implies the following

Theorem 1 (Serre's Conjecture). *If X is a simply connected finite complex with nontrivial reduced mod p homology, then the p-torsion subgroup of $\pi_n(X)$ is nonzero for infinitely many n.*

The second part of the seminar deals with the James construction and the James fibration sequence. We start by proving the Bott-Samelson theorem:

Theorem 2 (Bott-Samelson). *Let R be a principal ideal domain and X a connected space such that $H_*(X,R)$ is a free R-module. Then the homology ring $H_*(\Omega\Sigma X, R)$ (with the Pontryagin product) is isomorphic to the tensor algebra $T_R(\widetilde{H_*}(X,R))$.*

Using this theorem one can deduce that the James construction $J(X)$ is a model for $\Omega\Sigma X$. Next, we discuss Samelson products and construct the 2-local James fibration sequence

$$S^n \xrightarrow{\Sigma} \Omega S^{n+1} \xrightarrow{h} \Omega S^{2n+1}$$

which yields the 2-local EHP long exact sequence

$$\cdots \longrightarrow \pi_{n+k}(S^n) \xrightarrow{E} \pi_{n+k+1}(S^{n+1}) \xrightarrow{H} \pi_{n+k+1}(S^{2n+1}) \xrightarrow{P} \pi_{n+k-1}(S^n) \longrightarrow \cdots.$$

Finally, using the James fibration, we prove James's 2-primary exponent theorem:

Theorem 3 (James). *4^n annihilates the 2-primary component of $\pi_*(S^{2n+1})$.*

The last part of the seminar is devoted to the EHP spectral sequence. The EHP exact sequences assemble into an exact couple that gives the EHPSS

$$E_1^{kn} = \pi_{k+n}(S^{2n-1}) \Rightarrow \pi_k^S.$$

Here π_k^S, $k \geq 0$, denote the stable stems. The moral of the EHPSS is that it computes its input. For example, using the knowledge of the first five 2-local stable stems (the computations of lower dimensional stable stems was discussed in one of the previous terms), one can compute the unstable homotopy groups $\pi_5(S^3)$, $\pi_6(S^3)$, $\pi_7(S^4)$, $\pi_7(S^3)$, $\pi_8(S^4)$, and $\pi_9(S^5)$ 2-locally. We will go into details of these computations. Finally, in the last talk of the seminar, we will see the James periodicity theorem as well as connections with the J-homomorphism.

The main reference for the first two parts of the seminar is [3]. The references for the final part are [3] [4], [5], [7].

Talks

Talk 1 (**Homotopy groups with coefficients**, 12.04.2012-**Irakli Patchkoria**). Start with the definitions in [3, 1.1] (Definition 1.1.6 is especially important). Prove [3, 1.1.7, 1.1.9]. Only state the results of Exercises (1) and (2). Next, define the relative homotopy groups with coefficients and construct the long exact sequence as in [3, 1.2]. Finally, prove the universal coefficient theorem for mod k homotopy groups [3, 1.3.1]. (References: [3, 1.1-1.3].)

Talk 2 (**Bockstein sequence and mod k Hurewicz theorem**, 12.04.2012-**Irakli Patchkoria**). Start by discussing the functorial properties of $P^n(-)$ as in [3, 1.4]. Sketch the proof of [3, 1.4.2]. Next, construct the Bockstein sequence [3, 1.5]. Finally, define the mod k Hurewicz homomorphism and state the mod k Hurewicz theorem [3, 1.7-1.8] (which will be proved for simply connected spaces in Talk 6 using completions). If time permits, prove [3, Proposition 1.10.1] as an application. (References: [3, 1.4-1.10].)

Talk 3 (**Dror Farjoun-Bousfield localization**, 26.04.2012-**Uwe Kranz**). The objective of this talk is to introduce the general machinery of localization with respect to $M \longrightarrow *$, where M is a fixed pointed space. Start with the Definitions 2.1.1-2.1.3 of [3] and prove [3, 2.1.4-2.1.5]. Next, sketch the proof of [3, 2.1.6]. Finally prove the Zabrodsky lemma [3, 2.1.8] which is needed in Talk 7. (References: [3, 2.1].)

Talk 4 (**Inverting primes**, 26.04.2012-**Ruth Joachimi**). Introduce the notion of an S-local space, where S is a set of primes [3, 2.3.1] and prove [3, 2.3.4]. Next, apply the general results from Talk 3 and show that S-localizations exist and are unique up to homotopy. Further, prove [3, 2.3.5-2.3.6]. Only sketch the proof of [3, 2.7.3]. Finally, get the local Hurewicz theorem as a corollary [3, 2.3.14]. (References: [3, 2.3].)

Talk 5 (**Homological algebra background and p-completions of abelian groups**, 10.05.2012-**Antonio Sartori**). This talk is devoted to recall some algebraic constructions that are needed in later talks. The material is contained in [3, 2.4-2.6]. You can quickly review the constructions from [3, 2.4-2.5] as we assume that the audience is familiar with these things. However, you should take your time to discuss [3, 2.6] carefully. Define the notion of a p-complete abelian group [3, 2.6.1]. Prove [3, 2.6.4] and introduce p-completions [3, 2.6.5]. Next, prove [3, 2.6.6] which is the main result of this talk. Finally, prove [3, 2.6.8] and state its corollaries. (References: [3, 2.4-2.6].)

Talk 6 (**p-completions of simply connected spaces**, 10.05.2012-**Sebastian Thyssen**). Introduce the notion of a p-complete space [3, 2.7.1] and prove [3, 2.7.2-2.7.3] which yield the existence of p-completions. Next, discuss p-completions of Eilenberg-MacLane spaces [3, 2.7.4-2.7.6] and get [3, 2.7.7-2.7.8] as corollaries. As a nice application prove the mod k Hurewicz theorem for simply connected spaces [3, 2.8.1]. (References: [3, 2.7-2.8].)

Talk 7 (**Serre's theorem and Serre's conjecture**, 14.06.2012-**Irakli Patchkoria**). This talk is very challenging. Start with Lemma 2.10.1 [3] and recall the Moore-Postnikov decompositions. Next, prove [3, 2.10.2-2.10.5], state the theorem of Miller (the Sullivan conjecture for $\mathbb{Z}/p\mathbb{Z}$) and obtain Serre's theorem [3, 2.10.7] as a corollary. Finally, discuss the proof of Serre's conjecture [3, 2.10.8] using Miller's theorem. (References: [3, 2.10], [2].)

Talk 8 (**The Bott-Samelson theorem and the James construction**, 14.06.2012-**Anna Abczynski**). Start with the necessary definitions from [3, 4.1] and prove Proposition 4.1.2 [3]. Next, prove the Bott-Samelson theorem [3, 4.1.13]. Further, introduce the James construction $J(X)$ [3, 4.2] and, using the Bott-Samelson theorem, show that $J(X)$ is a model for $\Omega\Sigma X$ [3, 4.2.2]. As an amusing corollary we get the Freudenthal suspension theorem [3, 4.2.4]. (References: [3, 4.1-4.2].)

Talk 9 (**Samelson products and Hilton-Hopf invariants**, 28.06.2012-**Lennart Meier**). Define Samelson products as in [3, 4.3]. State Lemma 4.3.1 [3] and define the notion of a Hall basis. Next, state without a proof the expanded Hilton-Milnor theorem [3, 4.3.3]. Further, define Hilton-Hopf invariants and discuss their properties [3, 4.3.4-4.3.6]. (References: [3, 4.3].)

Talk 10 (**The EHP sequence and James's 2-primary exponent theorem**, 28.06.2012-**Karol Szumilo**). Introduce the generalized Hopf map h as in [3, 4.4]. Prove Corollary 4.4.3 [3] using [3, 4.4.1-4.4.2]. Next, prove [3, 4.4.4-4.4.5]. Further, state [3, 4.4.7] and get as a corollary the 2-local James fibration sequence [3, 4.4.7]. Altogether we obtain the EHP sequence at prime 2. Finally, using the results of the last talk prove [3, 4.5.1-4.5.2] and obtain the James 2-primary exponent theorem [3, Corollary 4.5.3 (c)]. (References: [3, 4.4-4.5].)

Talk 11 (**The EHP spectral sequence I**, 12.07.2012-**Karol Szumilo**). Construct the EHP spectral sequence as in [4, I.5]. Concentrate throughout on the 2-local case. Prove the basic facts on the EHP spectral sequence [4, 1.5.8]. Next, go through the computations on pages 26-28 in [4]. In particular, compute the differentials $d_1\colon E_1^{k,k+1} \longrightarrow E_1^{k-1,k}$, $k \geq 1$. For this you will need the description of the map P in terms of Whitehead products [7, Chapter XII, (2.4) Theorem], the computation at the end of [3, 4.4] and the connection between Samelson products and Whitehead products [3, 6.3]. Assuming that the we know the first five 2-local stable stems you should be able to compute 2-locally the unstable homotopy groups $\pi_5 S^3$, $\pi_6 S^3$ and $\pi_7 S^4$. Finally, sketch the 2-local computations of $\pi_7(S^3)$, $\pi_8(S^4)$, and $\pi_9(S^5)$ from [5, Chapter V]. (References: [4, I.5], [7, Chapter XII], [3, 4.4 and 6.3] and [5, V].)

Talk 12 (**The EHP spectral sequence II**, 12.07.2012-**Justin Noel**). Discuss the James periodicity theorem [4, 1.5.18] for the stable EHP spectral sequence of RP^∞ and connections with the J-homomorphism as in [4, I.5]. (References [4, I.5].)

References

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