HOMOTOPY & COHOMOLOGY

GK 1150

Young Women in Topology

Bonn, June 25 – 27, 2010

Factorability of Groups

Viktoriya Ozornova

Homology of Groups

The group homology is an algebraic invariant which enables the study of groups and, most notably, group actions. It allows to apply methods from other branches of mathematics, for example with topological methods or methods from homological algebra. (You can use 'cohomology' as well, dualizing all definitions.) The topological definition uses the 'classifying space' BG of a group G (where 'classifying' refers to G-principal bundles, and the space is well-defined up to homotopy equivalence), setting

$$H_*(G) := H_*(BG;\mathbb{Z}).$$

One also can add, as usual, coefficients, using any G-module, i.e. any abelian group with compatible G-action. For doing so, one assigns to a G-module M a local coefficient system $\mathcal{M}$ on BG which takes into account the G-action, and defines

$$H_*(G;M) := H_*(BG;\mathcal{M}).$$

(This definition reduces to the first one for the special case of G-module $\mathbb{Z}$ with trivial G-action.)

In some cases, the homotopy type of BG is represented by rather easy spaces: For example, $B(\mathbb{Z}^n) \simeq \mathbb{T}^n$, the n-dimensional torus, and the homology is easily computed. Yet, these spaces are in general rather complicated. For braid groups B_n, one can use the unordered configuration spaces $C_n(\mathbb{R}^2)$ as a model for a classifying space. Their cohomology with $\mathbb{Z}/2$-coefficients was computed by Fuks ([Fuks]) by means of an involved calculation. For the symmetric group $\mathfrak{S}_n$, the classifying space can be modelled by the unordered configuration space $C_n(\mathbb{R}^\infty)$ whose cohomology is still not described in a manageable form. (There are some results concerning the cohomology of symmetric groups, cf. e.g. [Adem].)

Thus, we may benefit from another definition of group homology. In homological algebra, one considers the category of G-modules as the abelian category of modules over the (non-commutative) group ring $\mathbb{Z}G$ and defines for any G-module M

$$H_*(G;M) := \mathrm{Tor}_*^{\mathbb{Z}G}(\mathbb{Z},M)$$

Thus, one needs for the computation of the homology a projective resolution of $\mathbb{Z}$ as a trivial $\mathbb{Z}G$-module. Sometimes, it is possible to find very convenient projective resolutions; this allows, for instance, a simple computation of homology of cyclic groups. In general, there is a 'standard' resolution that can be constructed for any group G, the so-called 'bar resolution'. We give here only the definition of the normalized bar complex in the bar-notation (i.e. the result of tensoring the bar resolution with $\mathbb{Z}$ over $\mathbb{Z}G$ since we're going to consider here only $\mathbb{Z}$-coefficients): the q-th group of this complex B_q is a free abelian group on the generators $[g_q|\ldots|g_1]$, which are 1 if at least one $g_i = 1$. The differential is given by $d = \sum(-1)^i d_i$ with

$$d_i([g_q|\ldots|g_1]) = \begin{cases} [g_q|\ldots|g_2] & \text{if } i = 0 \\ [g_q|\ldots|g_{i+1}g_i|\ldots|g_1] & \text{if } 0 < i < q \\ [g_{q-1}|\ldots|g_1] & \text{if } i = q \end{cases}$$

The Visy-Complex and Factorability Structure

In general, the modules of the bar resolution and the resulting bar complex are rather huge, so one tries to find smaller complexes to compute group homology. With this goal in mind, we consider some further structure on groups.
First, we deal with the notion of a 'norm' on a group: A function $N\colon G \to \mathbb{N}$ is a **norm** on G if for all $g, h \in G$, we have

- $N(g) = 0 \Leftrightarrow g = 1$.
- $N(g \cdot h) \le N(g) + N(h)$ (triangle inequality).
- $N(g^{-1}) = N(g)$ (symmetry).

As an example, each group carries the constant norm given by $N(g) = 1$ for all $g \in G \setminus \{1\}$. More interesting norms are word-length norms with respect to a given generating set (closed under inversion to ensure the symmetry), defined as the smallest number of generators needed to write a given element, counted with multiplicity.

Any norm N yields a filtration on the bar complex above by setting $N([g_q|\ldots|g_1]) = N(g_q) + \ldots + N(g_1)$ and considering all elements of norm $\le h$ as $\mathcal{F}_h B_q G$, the **norm filtration**. One obtains for each h a complex $\mathcal{N}_*(G)[h]$ with $\mathcal{N}_q(G)[h] = \mathcal{F}_h B_q G/\mathcal{F}_{h-1}B_q G$ and the induced differential. These complexes are part of a spectral sequence (of a filtered complex) which computes the homology of G. The following theorem simplifies this spectral sequence considerably and should motivate the definition of a 'factorability structure' on a normed group:

Theorem 1 (Bödigheimer, Visy) *If (G, N, η) is a factorable normed group, then the homology of $\mathcal{N}_*(G)[h]$ is concentrated in degree h.*

Loosely speaking, the factorability structure postulates the existence of a normal form, a preferred reduced expression (in elements of minimal norm) chosen. More precisely:

Definition 1 (Bödigheimer, Visy) *Let (G, N) be a normed group, and $T(G)$ the set of elements of minimal positive norm. A **factorability structure** is a function $\eta = (\overline{\eta}, \eta')\colon G \to G \times G$ with the following properties:*

(F1) For all $g \in G$, we have $g = \overline{\eta}(g)\eta'(g)$.

(F2) For all $g \in G$, we have $N(g) = N(\overline{\eta}(g)) + N(\eta'(g))$.

(F3) For all $g \in G \setminus \{1\}$, $\eta'(g) \in T(G)$.

Furthermore, η has to satisfy the following additional properties, which ensures connect the group multiplication with the factorability map: Write for short $\overline{\eta}(a) = \overline{a}$ and $\eta'(a) = a'$. Then η must satisfy for any $a \in G$ and $b \in T(G)$:

(F4) $N(\overline{ab}) + N((ab)') = N(a) + N(b) \Leftrightarrow N(\overline{a}\cdot\overline{a'b}) + N((a'b)') = N(a) + N(b)$.

(F5) $N(\overline{ab}) + N((ab)') = N(a) + N(b)$ implies $(\overline{ab}, (ab)') = (\overline{a}\cdot\overline{a'b}, (a'b)')$.

Some Examples and Coxeter Groups

Note that the factorability structure depends not only on the group itself but also on the norm chosen. Here are several examples:

1. Every group G is factorable with respect to the trivial norm via $\eta(g) = (1, g)$ for all $g \in G$. Note that in this case the theorem above is true for trivial reasons; the computation of the homology from the spectral sequence is just the computation of homology of bar complex.
2. For $\mathbb{Z} = \langle t \rangle$, we can consider the word length norm with respect to the generating system $\{t, t^{-1}\}$. This normed group is factorable in the obvious way.
3. On the other hand, if we consider cyclic group $\mathbb{Z}/m\mathbb{Z} = \langle t | t^m = 1\rangle$ with the word length norm with respect to the generating system $\{t, t^{-1}\}$, then this normed group does not admit a factorability structure for $m > 3$.
4. Factorability structures are quite well-behaved: a direct product as well as a free product of two factorable normed groups carries a natural norm and a factorability structure obtained from the ones of the factors. Furthermore, the semi-direct product also carries a factorability structure with respect to a naturally defined norm if the action defining the semi-direct product is norm-preserving.
5. On the symmetric group $\mathfrak{S}_n$, there are two generating systems of particular interest: the one of simple transpositions (i.e. of the form $(i, i+1)$) and the one of all transpositions. It can be shown that the symmetric group $\mathfrak{S}_n$ with the word length norm of the first system does not carry a factorability structure. B. Visy defines a factorability structure on $\mathfrak{S}_n$ with respect to the second norm ([Visy]).

As a generalization, one can explore the existence of (non-trivial) factorability structures on Coxeter groups and on braid groups.

A **Coxeter group** is a group which has a presentation with a set of generators S and only relations of the type

$$(st)^{m(s,t)} = 1$$

for all $s, t \in S$, where $m(s,s) = 1$, $m(s,t) = m(t,s) \ge 2$ for $s \ne t$ and the value ∞ ('unrelated') is allowed. Some examples of Coxeter groups are given by symmetric groups and dihedral groups (generated by simple reflections and two reflection, respectively). The Coxeter groups are of particular interest in this context since on the one hand, they are known to have good properties of word length with respect to S, also called 'Coxeter length', and on the other hand, the groups themselves are of interest, e.g. as all Weyl groups are finite Coxeter groups. By 'good properties', we mean here some theorems concerning the reduced words for an element. Those can be found in any textbook on Coxeter groups, e.g. [Davis].

Unfortunately, one can show that almost all Coxeter groups (those which have at least one relation as above with $2 < m(s,t) < \infty$) with the Coxeter length as a norm do not admit a factorability structure.

On the other hand, one can consider the generating system R of all conjugates of S, which are often called 'reflections'. In the case of symmetric groups, these are exactly the transpositions, and the corresponding normed group admits a factorability structure. For the dihedral groups with the word length norm with respect to all reflections, a factorability structure can also be constructed easily. Thus, it is quite natural to investigate whether every Coxeter group admits a factorability structure with respect to word length norm of reflections.

Outlook

It would be also interesting to see how the spectral sequence above for a norm admitting factorability structure looks like in the constructed examples. More generally, one can ask which norms allow a factorability structure that simplify the calculation of the group homology. Here, the considerations were restricted to $\mathbb{Z}$-coefficients for simplicity, but coefficients may also be involved in this context, after adding a 'norm' structure to them. Again, one can ask under which circumstances this structure makes the homology calculation easier.

A further point to consider is the question when homomorphisms of factorable groups exist. For example, the notion of factorability is not well-behaved under taking subgroups and quotients, yet one can try to find conditions under which subgroups and quotients have a factorability structure.
Furthermore, it could be studied whether some variations of the definition of factorability structure are as well appropriate to show the theorem above and whether these structures then are easier to construct.

We are also interested on the norm-filtration itself, in particular for the symmetric groups. Indeed, the geometric version of the bar complex is a possible construction for a classifying space, and we obtain thus a norm filtration of a classifying space. One can try to identify this filtration on the other model of the classifying space for the symmetric groups mentioned above: the unordered configuration space $C_n(\mathbb{R}^\infty)$. Furthermore, the homology of the filtration quotients (in the geometric version) builds the E_1-term of a spectral sequence converging to the homology of certain moduli spaces. One can hope to win some information from this rather unexpected connection.

References

[Adem] **A.Adem, R.Milgram**, *Cohomology of Finite Groups*, Springer-Verlag, 1994

[Davis] **M.Davis**, *The Geometry and Topology of Coxeter Groups*, London Mathematical Society Monographs, Princeton University Press, 2008

[Fuks] **D.Fuks**, *Cohomologies of the group cos mod 2*, Funct. Anal. Appl., **4**, (1970), pp. 143–151.

[Visy] **B.Visy**, *Homology of Normed Groups and of Graph Complexes, applied to Moduli Spaces*, thesis (in preparation)

Advisor: Prof. Dr. Bödigheimer
Universität Bonn