Young Women in Topology

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Symmetric squaring

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Introduction

What is meant by symmetric squaring?

Symmetric squaring is the operation of performing a squaring first and then dividing the result by a symmetric map. More precisely, let X be a topological space and define the coordinate-flipping involution τ by

$$\tau\colon X \times X \to X \times X$$
$$(x, y) \mapsto (y, x).$$

Then the topological space $X \times X/\tau$ is called the symmetric square of X and it will be denoted by $X^{\mathbf{s}}$. For pairs of topological spaces (X, A), the symmetric squaring is defined in an analogous manner as

$$(X, A)^{\mathbf{s}} := (pr(X \times X), pr(X \times A \cup A \times X \cup \Delta)),$$

where $pr\colon X \times X \to X \times X/\tau$ denotes the canonical projection and $\Delta := \{(x, x) | x \in X\} \subset X \times X$ denotes the diagonal in $X \times X$.

The diagonal is added to the subspace of $(X, A)^{\mathbf{s}}$ for technical reasons. Especially if X is a smooth manifold, it is necessary to cut the diagonal out or to at least work relative to it in homology. Since the involution τ leaves the diagonal fixed, the quotient by τ does not have a canonical smooth structure there. Outside the diagonal, however, there is a smooth structure which can and will be used to think of $X^{\mathbf{s}} \setminus pr(\Delta)$ as a smooth manifold whenever X is a smooth manifold.

What is symmetric squaring good for?

In [SSST10] the symmetric squaring construction was introduced and used in the context of Čech homology with $\mathbb{Z}_2$-coefficients and it was generalized to Čech homology with integer coefficients for even dimensions in [Nak07]. For a proof of the generalized Borsuk Ulam theorem in [SSST10], a non-trivial homology class has to be constructed. Symmetric squaring is a valuable tool for this purpose because it has the property of mapping the fundamental class of a manifold to the Čech homology version of the fundamental class of the symmetric square of the manifold.

The symmetric squaring as such is considered to be a construction of independent interest which is worth to be examined in other contexts. Here the centre of consideration is the question how the symmetric squaring induces well-defined maps in different settings as homology and bordism. It is ongoing work to generalize the symmetric squaring to the latter.

Is there a connection between symmetric squaring in homology and in bordism?

Ever since the concept of bordism was first introduced in [Tho54] there has also been known a natural transformation between bordism and homology. This transformation uses the existence of a unique fundamental class. As was already mentioned, the symmetric squaring construction behaves well with respect to fundamental classes. Putting these facts together it can be proven that the fundamental class transformation between bordism and homology is compatible with the symmetric squaring. This means that passing first from bordism to homology and performing the symmetric squaring then is the same map as using the symmetric squaring first in bordism and passing to homology afterwards.

Symmetric squaring in bordism

Singular bordism

See [CF62] to learn more about singular bordism.

Definition 1 ((Un)oriented singular bordism) *Let (X, A) be a topological pair. A smooth compact oriented bounded n-manifold $(M, \partial M)$ together with a map $f\colon (M, \partial M) \to (X, A)$ is called a singular n-manifold in (X, A). Such a $(M, \partial M; f)$ is said to bord iff there exists $F\colon B \to X$ which satisfies*

- *B is a compact oriented $(n + 1)$-manifold with boundary,*
- *∂B contains M as a regular submanifold whose orientation is induced by that on B,*
- *F restricted to M is equal to f and $F(\partial B \setminus M) \subset A$.*

The oriented singular manifolds $(M_0, \partial M_0; f_0)$ and $(M_1, \partial M_1; f_1)$ are called bordant iff the disjoint union $(M_0 \sqcup -M_1, \partial(M_0 \sqcup -M_1); f_0 \sqcup f_1)$ bords. Unoriented bordism is defined similarly by dropping the orientability assumptions. "Bordant" is an equivalence relation and the sets of equivalence classes are called $\Omega_n(X, A)$ and $\mathcal{N}_n(X, A)$ respectively.

Čech bordism and the symmetric squaring in bordism

The first idea to transport symmetric squaring to bordism surely is to just perform the symmetric squaring on the manifolds itself. But the quotient by τ lacks a canonical smooth structure on the diagonal. There has to be found a way to remove the diagonal from the symmetric square, so that the result is a smooth compact manifolds again. It is not desirable, however, to impose a wide range of choices of subtracting the diagonal into our definition. The way out is to look at all such neighbourhoods at the same time:

Definition 2 (Čech bordism) *Following the definition of Čech homology as an inverse limit of singular homology groups, Čech bordism is defined as a certain inverse limit of relative bordism groups. More precisely, consider the neighbourhoods $V \subset Y$ of the subspace $B \subset Y$ in a topological pair (Y, B) as a quasi-ordered set ordered by inverse inclusion. Then the Čech bordism group of the pair (Y, B) is defined to be the inverse limit of the relative bordism groups of (Y, V) over this quasi-ordered set. Here, this leads to*

$$\check{\Omega}_n(X, A)^{\mathbf{s}} := \varprojlim_{U \supset \Delta} \left\{\Omega_n(X^{\mathbf{s}}, \overline{pr(X \times A \cup A \times X \cup U)})\right\}.$$

Definition 3 (symmetric squaring in bordism) *Let (X, A) be a pair of topological spaces. The symmetric squaring in bordism is defined as*

$$(\;\cdot\;)^{\mathbf{s}}\colon \Omega_k(X, A) \to \check{\Omega}_{2k}((X, A)^{\mathbf{s}})$$
$$[M, \partial M; f] \mapsto [M, \partial M; f]^{\mathbf{s}} := \left\{\left[(M \times M \setminus U)/\tau, \partial(-), f \times f/\tau_{|(-)}\right]\right\}_{U \supset \Delta}.$$

It is a rather technical but necessary step to restrict the above limit to those neighbourhoods U of the diagonal that have a smooth compact bounded manifold $M \times M \setminus U$ as their complement. But it is possible and does not change the above definition much.

Symmetric squaring in homology and results

Definition 4 (symmetric squaring in homology) *Denote the k-th singular chain group of the topological pair (X, A) by $C_k(X, A, \mathbb{Z})$. Define*

$$(\;\cdot\;)^{\mathbf{s}}\colon C_k(X, A, \mathbb{Z}) \to C_{2k}((X, A)^{\mathbf{s}}, \mathbb{Z}) \text{ by}$$
$$\sigma = \sum_{i=1}^{n} g_i \sigma_i \mapsto \sigma^{\mathbf{s}} := \sum_{\substack{i<j \\ 1 \le i,j \le n}} g_i g_j (pr)_\sharp(\sigma_i \times \sigma_j),$$

where $\times$ denotes the simplicial cross product and $(pr^X)_\sharp$ is the chain map induced by the projection $pr\colon X \times X \to X \times X/\tau$.

This induces a symmetric squaring map in homology, so that symmetric squaring maps have been defined in bordism and in homology now. It can be proven that these mappings are well-defined in many cases. The map τ being an orientation reversing map in odd dimensions limits considerations to even dimensions while dealing with $\mathbb{Z}$-coefficients or oriented manifolds respectively, since there is no canonical orientation on the quotient by τ in these cases.

Theorem 1 (symmetric squaring is well-defined) *Definitions 4 and 3 give well-defined maps in all dimensions k in unoriented (Čech) bordism and in (Čech) homology with $\mathbb{Z}_2$-coefficients as well as in all even dimensions k in oriented (Čech) bordism and in (Čech) homology with $\mathbb{Z}$-coefficients.*

This well-defined map "maps fundamental classes to fundamental classes":

Theorem 2 (behaviour with respect to fundamental classes) *Let $(B, \partial B)$ be a k-dimensional compact smooth oriented manifold and let $\sigma_{\mathbf{f}} \in H_k(B, \partial B, \mathbb{Z})$ be its fundamental class. Then $\sigma_{\mathbf{f}}^{\mathbf{s}} \in \check{H}_{2k}((B, \partial B)^{\mathbf{s}})$ is the fundamental class of $(B, \partial B)^{\mathbf{s}}$ in the following sense. For every neighbourhood U of the diagonal in $B \times B$, there is a fundamental class $\sigma_{\mathbf{f}}^U \in H_{2k}\left(((B \times B) \setminus (\partial(B \times B) \cup U))/\tau, \partial(-), \mathbb{Z}\right)$ which can be mapped by inclusion to $i(\sigma_{\mathbf{f}}^U) \in H_{2k}\left(B^{\mathbf{s}}, \overline{pr(\partial(B \times B) \cup U)}, \mathbb{Z}\right)$. It is true that $p(\sigma_{\mathbf{f}}^{\mathbf{s}}) = i(\sigma_{\mathbf{f}}^U)$, where p denotes the projection onto the factor of U in the inverse limit group $\check{H}_{2k}((B, \partial B)^{\mathbf{s}})$.*

Definition 5 (fundamental class transformation) *A passage from bordism to homology can be defined in the following way:*

$$\mu\colon \Omega_k(X, A) \to H_k(X, A, \mathbb{Z})$$
$$[M, \partial M; f] \mapsto \mu(M, \partial M, f) := H_k(f)(\sigma_{\mathbf{f}}),$$

where $\sigma_{\mathbf{f}} \in H_k(M, \partial M, \mathbb{Z})$ is the fundamental class and $H_k(f)$ is the map which is induced by $f\colon (M, \partial M) \to (X, A)$ in homology. This map was first introduced in [Tho54].

Compatibility and Čech versions vs. ordinary versions

Putting all given facts together, the following theorem results.

Theorem 3 (compatibility) *Let $k \in \mathbb{N}$ be even and (X, A) a topological pair. Then the diagram*

$$\begin{array}{ccc}
\Omega_k(X, A) & \xrightarrow{\mu} & H_k(X, A, \mathbb{Z}) \\
\downarrow{\scriptstyle (\cdot)^{\mathbf{s}}} & & \downarrow{\scriptstyle (\cdot)^{\mathbf{s}}} \\
\check{\Omega}_{2k}((X, A)^{\mathbf{s}}) & \xrightarrow[\check{\mu}]{} & \check{H}_{2k}((X, A)^{\mathbf{s}}, \mathbb{Z}),
\end{array}$$

is commutative.

Using Čech homology and bordism here has been important for the intuition and the proofs. But it can be proven that in a lot of interesting cases, the ordinary groups are isomorphic. The statement concerning homology in the following theorem has been shown in [Dol80], proposition 13.17.

Theorem 4 (isomorphy of singular and Čech versions) *Let (X, A) be such that X is an ENR and $A \subset X$ is an ENR as well. Then*

$$\check{\Omega}_*(X, A) \simeq \Omega_*(X, A) \text{ and}$$
$$\check{H}_*(X, A) \simeq H_*(X, A, \mathbb{Z}).$$

References

[CF62] **P. E. Conner, E. E. Floyd**, *Differentiable periodic maps*, Bull. Amer. Math. Soc. **68**, (1962), 76–86.

[Dol80] **A. Dold**, *Lectures on algebraic topology.*, Second Edition, Springer-Verlag, Berlin, **200**, Grundlehren der Mathematischen Wissenschaften, (1980), xi+377.

[Nak07] **D. Nakiboğlu**, *Die Dachabbildung in ganzzahliger Čech-Homologie*, diploma thesis, http://arxiv.org/pdf/1002.1449.

[SSST10] **T. Schick, R. Simon, S. Spiez, H. Torunczyk**, *A parametrized version of the Borsuk Ulam theorem*, preprint, http://arxiv.org/pdf/0709.1774v4.

[Tho54] **R. Thom**, *Quelques propriétés globales des variétés différentiables.*, Comment. Math. Helv. **28**, (1954), 17–86.

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