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Constructing Generalised Leray Spectral Sequences

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Motivation - Construction of the First Spectral Sequence

This project arose as a spin-off of an earlier one (with the same collaborators) to extend to some categories of stacks several classical results from geometry and topology.

We were led in particular to the question of defining generalised cohomology theories for topological stacks, which in turn raised the issue of a descent spectral sequence for sheaves of spectra on this context.

In this poster we therefore consider relevant notions of spectral sequences which we expect to be applied not just in our original project, but to have many other applications.

The First Spectral Sequence

If $\mathcal{C}$ is a small category then $F\mathcal{C}$ is the category of factorisations, with objects the morphisms f in $\mathcal{C}$ and morphisms $f \to g$ the pairs (a, b) such that $bfa = g$. Recall [BW, Theorem 4.4] the Baues-Wirsching cohomology of $\mathcal{C}$ with coefficients $D : F\mathcal{C} \to Ab$ (a 'natural system') can be identified as

$$H^n_{BW}(\mathcal{C}, D) \;\cong\; \mathrm{Ext}^n_{F\mathcal{C}}(\mathbb{Z}, D) \;\cong\; \lim_{F\mathcal{C}}{}^n D \;\cong\; Ran^n_{(F\mathcal{C}\to *)}(D),$$

where $\mathbb{Z} : F\mathcal{C} \to Ab$ is the constant functor.

Theorem 1 (Leray-type spectral sequence for Baues-Wirsching cohomology) *Let $\mathcal{E}$ and $\mathcal{B}$ be small categories and $u : \mathcal{E} \to \mathcal{B}$ be a functor. Given a natural system $D : F\mathcal{E} \to Ab$ on $\mathcal{E}$, there is a first quadrant cohomology spectral sequence*

$$E_2^{p,q} \cong H^p_{BW}(\mathcal{B}, (R^q Fu_*)(D)) \Rightarrow H^{p+q}_{BW}(\mathcal{E}, D)$$

which is functorial with respect to natural transformations, where $R^q Fu_* = Ran^q_{Fu}$ is the q-th right satellite of the right Kan extension Ran_{Fu} along Fu.

Proof: We get an induced natural system $Fu_*(D)$ on D via the direct image or push forward functor

$$[F\mathcal{E}, Ab] \xrightarrow{Fu_*} [F\mathcal{B}, Ab]$$

which is right adjoint to the inverse image or pullback functor

$$[F\mathcal{B}, Ab] \xrightarrow{Fu^*} [F\mathcal{E}, Ab].$$

In this situation we get the desired first quadrant spectral sequence from [An], [Hu, Remark 2.2],

$$E_2^{p,q} \cong H^p_{BW}(\mathcal{B}, Ran^q_{Fu}(D)) \Rightarrow H^{p+q}_{BW}(\mathcal{E}, D)$$

□

Variations on the theme

Baues–Wirsching cohomology is in a sense a generalisation of various other notions of cohomology, such as Hochschild–Mitchell cohomology, or cohomology with local coefficients. In general, consider

$$F\mathcal{E} \xrightarrow{Fu} F\mathcal{B} \longrightarrow \mathcal{B}^{op} \times \mathcal{B} \longrightarrow \mathcal{B} \longrightarrow \pi_1\mathcal{B}$$

and write $\mathcal{C}$ for any of the categories in this diagram and $u' : F\mathcal{E} \to \mathcal{C}$ for the composite functor.
As in the previous theorem we have in each case a diagram

$$\begin{tikzcd}
{[F\mathcal{E}, Ab]} \arrow[rr, shift right, "u'_*"'] \arrow[dr, shift left, "\lim_{F\mathcal{E}}"] & & {[\mathcal{C}, Ab]} \arrow[ll, shift right, "u'^*"'] \arrow[dl, shift left, "\lim_{\mathcal{C}}"] \\
& Ab \arrow[ul, shift left, "c"] \arrow[ur, shift left, "c"] &
\end{tikzcd}$$

where c denotes the constant diagram functors, u'^* is precomposition with u', and the other functors in the diagram are the right adjoints of these, given by the limits $\lim_{F\mathcal{E}}$, $\lim_{\mathcal{C}}$ and by $u'_* = Ran_{u'}$.
The spectral sequence for the derived functors of the composite

$$\lim_{F\mathcal{E}}(-) = \lim_{\mathcal{C}} u'_*(-)$$

converges to the Baues–Wirsching cohomology of $\mathcal{E}$ with coefficients in $D \in [F\mathcal{E}, Ab]$,

$$E_2^{p,q} \cong H^p(\mathcal{C}, Ran^q_{u'}(D)) \Rightarrow H^{p+q}_{BW}(\mathcal{E}, D).$$

In certain cases the Kan extension and the E_2 term can be identified explicitly and simplified:

Theorem 2 (Case $\mathcal{C} = F\mathcal{B}$, $u' = Fu$) *Given a functor $u : \mathcal{E} \to \mathcal{B}$ and a natural system $D : F\mathcal{E} \to Ab$, there is a first quadrant cohomology spectral sequence*

$$E_2^{p,q} \cong H^p_{BW}(\mathcal{B}, \lim_{\beta/Fu}{}^q D \circ Q^*) \Rightarrow H^{p+q}_{BW}(\mathcal{E}, D)$$

where $Q^* : \beta/Fu \to F\mathcal{E}$ is the forgetful functor.

Theorem 3 (Case $\mathcal{C} = \mathcal{B}$, $u' : F\mathcal{E} \to \mathcal{B}$) *Given functors $u : \mathcal{E} \to \mathcal{B}$ and $u' : F\mathcal{E} \to \mathcal{B}$ as above, and $\beta : b \to b'$ an object of the factorization category $F\mathcal{B}$, there is an isomorphism of categories $\beta/u' \cong F(b'/u)$. Given a natural system $D : F\mathcal{E} \to Ab$ on $\mathcal{E}$, there is a first quadrant cohomology spectral sequence*

$$E_2^{p,q} \cong H^p(\mathcal{B}, H^q_{BW}(-/u, D(-)) \Rightarrow H^{p+q}_{BW}(\mathcal{E}, D)$$

Grothendieck and Kan fibrations

Definition 1 [SGA] *Let $\mathcal{E}$ and $\mathcal{B}$ be small categories. A Grothendieck fibration is a functor $u : \mathcal{E} \to \mathcal{B}$ such that the fibers $\mathcal{E}_b$ depend contravariantly and pseudofunctorially on the objects b of the category $\mathcal{B}$. The category $\mathcal{E}$ is then also called a category fibred over $\mathcal{B}$.*

There are many equivalent explicit definitions of Grothendieck fibration. For our purposes we recall from [Gr1] that $u : \mathcal{E} \to \mathcal{B}$ is a Grothendieck fibration if for each object b of $\mathcal{B}$ the obvious inclusion functor between the fibre and comma categories

$$j_b : \mathcal{E}_b \to u/b, \quad e \mapsto (e, ue \xrightarrow{=} b)$$

is coreflexive, that is, has a right adjoint left inverse.
Associated to each Grothendieck fibration $u : \mathcal{E} \to \mathcal{B}$ is a pseudofunctor

$$\mathcal{B}^{op} \to CAT, \qquad b \mapsto u^{-1}(b)$$

which defines an equivalence between the 2-category of Grothendieck fibrations over a category $\mathcal{B}$ and that of contravariant pseudofunctors from $\mathcal{B}$ to the category CAT of small categories,

$$Fib(\mathcal{B}) \overset{\simeq}{\leftrightarrows} [\mathcal{B}^{op}, CAT]$$

The inverse equivalence is given by the Grothendieck construction

$$\int : [\mathcal{B}^{op}, CAT] \to Fib(\mathcal{B}), \qquad u \mapsto \int_{\mathcal{B}} u.$$

The following can thus be considered as a translation of a result of [PR].

Theorem 4 *Let $\mathcal{E}$ and $\mathcal{B}$ be small categories and $u : \mathcal{E} \to \mathcal{B}$ be a Grothendieck fibration. Given a natural system $D : F\mathcal{E}^{op} \to Ab$ on $\mathcal{E}$, there is a first quadrant spectral sequence*

$$E_2^{p,q} \cong H^p(\mathcal{B}, \mathcal{H}^q_{BW}(u/-, D_{(-)})) \Rightarrow H^{p+q}_{BW}(\mathcal{E}, D)$$

which is functorial with respect to 1-morphisms of Grothendieck fibrations.

We also consider the case of Kan fibrations $K \to L$ between simplicial sets and, by considering the induced functors on the category of simplices $u : \Delta/X \to \Delta/Y$, obtain Serre type spectral sequences.

Generalised cohomology theories. . . ?

Definition 2 *Let $\mathcal{M} = Sp$ be the closed model category of fibrant spectra and let $\mathcal{L} : \mathcal{E} \to \mathcal{M}$ be a functor. The nth generalised cohomology group of the category $\mathcal{E}$ with coefficient system $\mathcal{L}$ is defined as:*

$$H^n(\mathcal{E}, \mathcal{L}) = \pi_n \mathbb{R}p_{\mathcal{E}*}\mathcal{L} = \pi_n \mathrm{holim}_{\mathcal{E}}\mathcal{L}$$

Here $\mathbb{R}p_{\mathcal{E}*} : \mathrm{Ho}([\mathcal{E}, \mathcal{M}]) \to Ho(\mathcal{M})$ is right derived functor of the constant diagram functor.

Theorem 5 (Generalised Leray spectral sequence) *For $u : \mathcal{E} \to \mathcal{B}$ and $\mathcal{L} : \mathcal{E} \to \mathcal{M}$ we have*

$$E_2^{p,q} \cong H^p(\mathcal{B}, \pi_q \mathbb{R}u_*\mathcal{L}) \Rightarrow H^{p+q}(\mathcal{E}, \mathcal{L}) \qquad (or: \quad E_2^{p,q} \cong \lim^p_{\mathcal{B}} \pi_q \mathbb{R}u_*\mathcal{L} \Rightarrow \pi_{p+q}\mathrm{holim}_{\mathcal{E}}\mathcal{L}.)$$

This follows from a kind of Grothendieck spectral sequence for the composition of right derived functors,

$$E_2^{p,q} \cong \pi_p \mathbb{R}p_{\mathcal{B}*} \circ \pi_q \mathbb{R}u_*(\mathcal{L}) \Rightarrow \pi_{p+q}\mathbb{R}p_{\mathcal{E}*}(\mathcal{L}).$$

Thomason's spectral sequence [Th] arises as an absolute case: taking $u = 1_{\mathcal{E}}$ we have

$$E_2^{p,q} \cong H^p(\mathcal{E}, \pi_q\mathcal{L}) \Rightarrow H^{p+q}(\mathcal{E}, \mathcal{L}).$$

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