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Universal Toda brackets of Commutative Ring Spectra

Boryana Dimitrova

The Project

Let A be a differential graded algebra. One can form the *Hochschild cohomology groups* $\mathrm{HH}^{*,*}(H^*A)$ of the cohomology algebra H^*A of A (see the right-hand side for a precise definition). In [BKS] Benson, Krause and Schwede construct a *canonical class*

$$\gamma_A \in \mathrm{HH}^{3,-1}(H^*A)$$

for an arbitrary differential graded algebra A over a field k. This class is the first piece of information of the A^∞*-structure* of H^*A, given by a theorem of Kadeishvili in [Ka1]. An interesting property of this canonical class is that it determines all of the so called *triple Massey products* of A. These are higher order generalized cohomology operations introduced by William Massey in the late sixties.

If in addition we take A to be commutative and k of characteristic 0, we can construct the A^∞-structure of H^*A in such a way that it is even C^∞ (also known as balanced or commutative A^∞-algebra) - for a reference see for example [Ma]. A consequence of this fact is that the canonical class γ_A can be viewed as an element in not just the Hochschild cohomology but also in the *Harrison cohomology* of A. By these means we get relations for the Massey products of A, e.g. the Leibniz rule:

$$0 \in < a,b,c > +(-1)^{|a||b|+|a||c|} < b,c,a > +(-1)^{|c||a|+|c||b|} < c,a,b >$$

for $a,b,c \in H^*A$ with $ab = 0 = bc$, but also many others induced by the *shuffle product*.

In the topological context *Toda brackets* are the corresponding equivalent to the Massey products in algebra. In fact, Toda brackets can be defined in any triangulated category and Massey products in the derived category of a differential graded algebra A coincide with the Toda brackets.

For an associative ring spectrum R we can form the category of R-module spectra. This has the structure of a stable model category, and so we can build its homotopy category $\mathcal{D}(R)$, which is in fact triangulated. In his PhD thesis Steffen Sagave [Sa] constructs for an arbitrary (associative) ring spectrum R a *universal Toda bracket* γ_R

$$\gamma_R \in \mathrm{HML}^{3,-1}(\pi_*(R)),$$

which lives in the MacLane cohomology of the homotopy ring of R and also recovers all triple *matric* Toda brackets.

Question: Is there an equivalent to Harrison cohomology if we take R to be a *commutative* ring spectrum? In other words, is there an appropriate cohomology theory, in which the universal Toda bracket of R lives, and what kind of additional relations do Toda brackets satisfy in the commutative case?

Definitions

Definition 1 (triple Massey products) *Let A be a differential graded algebra and $a,b,c \in H^*A$ homogeneous elements in the cohomology of A such that $ab = 0 = bc$. The Massey product $< a,b,c >$ is a subset of $H^{|a|+|b|+|c|-1}A$ given as follows: take representative cocycles of a,b,c in A - say u,v,w, respectively. Now, there are s and t in A, such that $uv = d(s)$ and $vw = d(t)$. A short calculation shows that $sw + (-1)^{|u|+1}ut$ is a cocycle. We define the set of cohomology classes of all the elements obtained in this way (considering all the possible choices for u,v,w,s,t) to be the triple Massey product $< a,b,c >$.*

Remark 1 *$< a,b,c >$ is a coset of $H^{|a|+|b|+|c|-1}A$ for the subgroup $a \cdot (H^{|b|+|c|-1}A) + (H^{|a|+|b|-1}A) \cdot c$.*

Definition 2 (triple Toda bracket) *Let $\mathcal{T}$ be a triangulated category and a,b and c three composable maps in $\mathcal{T}$ such that $ab = 0$ and $bc = 0$. The Toda bracket $< a,b,c >$ is a subset of the group of morphisms $\mathcal{T}(\Sigma W, Z)$, given by the following construction: Choose a distinguished triangle starting with c as in the upper row of the following diagram.*

$$\begin{tikzcd}
W \arrow[r, "c"] & X \arrow[r] \arrow[dr, "b"] & C(c) \arrow[r] \arrow[d, dashed, "b'"] & \Sigma W \arrow[d, dashed, "f \in <a,b,c>"] \\
 & & Y \arrow[r, "a"] & Z
\end{tikzcd}$$

Since the compositions ab and bc are trivial, we can find maps b' and f as above making everything commute. The elements of the Toda bracket are all the maps $\Sigma W \to Z$ that arise in this way (also considering different choices for b').

Remark 2 *$< a,b,c >$ is a coset of $\mathcal{T}(\Sigma W, Z)$ for the subgroup $a \circ \mathcal{T}(\Sigma W, Y) + \mathcal{T}(\Sigma X, Z) \circ \Sigma c$.*

Remark 3 *There are also definitions of* higher Massey products *as well as* higher Toda brackets.

Definition 3 (Hochschild cohomology) *Let A be a graded k-algebra with commutative ground ring k and M a graded A-bimodule. Denote by $Hom_k^m(A^{\otimes n}, M) =: C^{n,m}(A,M)$ the set of graded k-linear maps from $A^{\otimes n}$ to M that raise the degree by m. There is a differential $\delta\colon C^{n,m}(A,M) \to C^{n+1,m}(A,M)$ given by*

$$\begin{aligned}\delta f(a_1,\ldots,a_{n+1}) = \quad & (-1)^{m|a_1|}a_1 f(a_2,\ldots,a_{n+1}) + \textstyle\sum_{k=1}^n (-1)^k f(a_1,\ldots,a_k a_{k+1},\ldots,a_{n+1}) \\ & +(-1)^{n+1} f(a_1,\ldots,a_n)a_{n+1}\end{aligned}$$

that makes $(C^{*,*}(A,M),\delta)$ into a cochain complex called the Hochschild cochain complex of A with coefficients in M. The cohomology groups of this complex give the Hochschild cohomology $\mathrm{HH}^{*,*}(A,M)$ of A with coefficients in M.

Remark 4 *For $M = A$ we denote by $\mathrm{HH}^{*,*}(A) \colon = \mathrm{HH}^{*,*}(A,A)$ the Hochschild cohomology of A.*

Approach

The idea is to try to 'mimic' the algebraic case as far as possible. To be more precise, let us roughly recall Harrison cohomology:

The *Harrison complex* $\overline{C}^{*,*}(A,A)$ is a subcomplex of the Hochschild complex $C^{*,*}(A,A)$, that consists of exactly those maps $f \in C^{*,*}(A,A)$ that vanish on the so called *shuffle products*. For example if we take $f \in C^{3,-1}(A,A)$, then f is an element of the Harrison complex if and only if

$$f(a \otimes b \otimes c + (-1)^{1+|a||b|} b \otimes a \otimes c + (-1)^{|a||b|+|a||c|} b \otimes c \otimes a) = 0$$

for all $a,b,c \in A$. In higher degrees there are more relations. The Harrison cohomology is the cohomology of the Harrison complex.

As mentioned earlier, the canonical class γ_A of a differential graded algebra A detects all triple Massey products. This means that if we take a representing cocycle $m \in C^{3,-1}(H^*A, H^*A)$ of γ_A, then $m(a \otimes b \otimes c)$ is an element of the Massey product $< a,b,c >$ (whenever this is defined). If in addition A is commutative, then a representing cocycle of γ_A can be chosen to be in the Harrison complex $\overline{C}^{3,-1}(H^*A, H^*A)$. In particular we get that

$$(\star) \quad 0 \in < a,b,c > \pm < b,a,c > \pm < b,c,a > .$$

This relation implies among others the Leibniz rule stated before.

On the topological side, the universal Toda bracket constructed by Sagave lives in the MacLane cohomology $\mathrm{HML}^{*,-\bullet}(\pi_*(R))$ of $\pi_*(R)$. This is the cohomology of the cochain complex $\widetilde{C}^*(\mathcal{F}(R), [-,-]_\bullet)$ given by

$$\widetilde{C}^s(\mathcal{F}(R), [-,-]_t) \colon = \{ m : N_s(\mathcal{F}(R)) \to \coprod_{(g: X_g \to Y_g) \in Mor(\mathcal{F}(R))} [X_g, Y_g]_t \mid m(g_1,\ldots,g_s) \in [X_{g_s \cdots g_1}, Y_{g_s \cdots g_1}]_t \}$$

for $s \geq 1$, where $\mathcal{F}(R)$ is a (small) full subcategory of $\mathcal{D}(R)$ consisting of the free modules, $N_s(\mathcal{F}(R))$ is the s-th level of the simplicial nerve of $\mathcal{F}(R)$, and $[X_g, Y_g]_t$ denotes the morphism space $\mathcal{D}(R)(X_g[t], Y_g)$. Together with an appropriate differential

$$\delta \colon \widetilde{C}^*(\mathcal{F}(R), [-,-]_\bullet) \to \widetilde{C}^{*+1}(\mathcal{F}(R), [-,-]_\bullet)$$

this gives a cochain complex.

Again as in the algebraic case, if m is a (normalized) cocycle representing γ_R then $m(g_1,g_2,g_3)$ is an element of the Toda bracket $< g_1,g_2,g_3 >$.

So a possible approach to the problem of constructing a new cohomology theory in which the universal Toda bracket of a commutative ring spectrum naturally lives, is trying to find a suitable subcomplex of the MacLane complex. For this it will be helpful to find relations for Toda brackets (triple and higher), generalizing the ones for Massey products. It seems to be rather unlikely for such relations to be as 'simple' as in the algebraic case described above, in particular because the C^∞-structure is only the right setting if the characteristic of the ground ring is zero. An intermediate step would be to look at the relations satisfied by Massey products of commutative differential graded algebras of positive characteristic.

Examples of Relations for Toda Brackets

Let R be a commutative ring spectrum. The following relations for triple Toda brackets hold:

$$\begin{aligned}
&0 \in < g_1,g_2,g_3 > +(-1)^{|g_1||g_2|+|g_1||g_3|} < g_2,g_3,g_1 > +(-1)^{|g_3||g_2|+|g_3||g_1|} < g_3,g_1,g_2 > && g_i \in \pi_*(R) \\
&0 \in < g_1,g_2,g_3 > +(-1)^{|g_1||g_2|+|g_1||g_3|+|g_2||g_3|} < g_3,g_2,g_1 > && g_i \in \pi_*(R)
\end{aligned}$$

Note that these two relations immediately imply the relation $(\star)$. Actually they are equivalent to it.

The above relations seem to generalize to higher Toda brackets, where we have:

$$\begin{aligned}
&0 \in < g_1,\ldots,g_n > \pm < g_2,g_3,\ldots,g_n,g_1 > \pm < g_3,\ldots,g_n,g_1,g_2 > \pm \cdots \pm < g_n,g_1,\ldots,g_{n-1} > \\
&0 \in < g_1,g_2,\ldots,g_n > \pm < g_n,g_{n-1},\ldots,g_1 >
\end{aligned}$$

To show this one has to use the construction of the universal Toda bracket of Sagave [Sa] as well as the concept of commutative J-space monoids.

For $n > 3$ the last two relations also follow from the shuffle product formulas (and thus are in particular true for higher Massey products provided the characteristic of the ground ring is 0). However, the converse does not hold: we cannot recover all the relations the shuffles imply just with those.

References

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Universität Bonn