HOMOTOPY & COHOMOLOGY

GK 1150

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1- dimensional EFT's

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talk 9
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Plan of the talk:

1. Rep.: 1- dim. EFT's
2. 1- dim. EFT's of degree n
3. Standard example of 1- dim. EFT_n's
4. 1- dim. susy EFT's (plain version)
5. GSM- categories and GSM- functors
6. 1- dim. susy EFT's (enriched version)
7. Standard example of 1-dim. susy EFT's
8. Degree datum for susy EFT's

① Rep.: 1-dim. EFT's

$\mathcal{EB}^1$: obj.: 0-spin mfds Y, i.e.

$$Y = \coprod_n pt \coprod_m \overline{pt}$$

$$\mathbb{R} \oplus 0 \to \mathbb{R}^0 \qquad 0 \oplus \mathbb{R} \to \mathbb{R}^0$$

morph.: 1) spin diffeom. $f: Y_1 \to Y_2$

2) 1-spin bordisms $\Sigma: Y_1 \to Y_2$ with Riem. metric, i.e. Σ 1-spin mfd with Riem. metric and $\partial\Sigma \cong \overline{Y_1} \sqcup Y_2$

$$\Sigma_1 = \Sigma_2 \Leftrightarrow \Sigma_1 \cong \Sigma_2 \text{ rel. bdry}$$

3 additional structures:

- symm. monoidal str.: disj. union of mfds
- involutions $\overline{(\ )}$ and anti-involutions $(\ )^*$:
  $\overline{(\ )}$ = taking the opposite spin str. on mfds and bordisms
  $(\ )^*$ = taking the opposite spin str. on bordisms (interpreted as reversal of direction)
- adjunction transformation:
  $\mathcal{EB}^1(\emptyset, \overline{Y_1} \sqcup Y_2) \to \mathcal{EB}^1(Y_1, Y_2)$

<u>Remark:</u> All morphisms in EB^1 are gen. by

1) kan. spin involution ε on pt

2) standard intervals of length t,
   $I_t : pt \to pt$

using comp., disj. union, inv. and adj. transf.

<u>in particular:</u>

- $\emptyset \xrightarrow{I_t} \overline{pt} \amalg pt \xrightarrow{I_{t'}} \emptyset$
  $\rightsquigarrow$ circle of length $t+t'$ with anti-per. spin str.

- $\emptyset \xrightarrow{I_t} \overline{pt} \amalg pt \xrightarrow{\varepsilon \amalg 1} \overline{pt} \amalg pt \xrightarrow{I_{t'}} \emptyset$
  $\rightsquigarrow$ circle of length $t+t'$ with periodic spin str.

! $\rightsquigarrow$ <u>moduli space of 1-spin bordisms</u> $= \mathbb{R}_{>0}$

$\mathcal{HS}$: <u>obj.:</u> sep. Hilbert spaces H

<u>morph.:</u> bounded operators $A: H_1 \to H_2$

<u>3 additional structures</u>:

- symm. monoidal str.: tensor product of Hilbert spaces
- involutions $\overline{(\ )}$ and anti-involutions $(\ )^*$:
  $\overline{(\ )}$ = taking the opposite complex str. on Hilbert spaces
  $(\ )^*$ – taking the adjoint on operators
- adjunction transformation:
  $\mathcal{HS}(\mathbb{C}, \overline{H}_1 \otimes H_2) \longrightarrow \mathcal{HS}(H_1, H_2)$

<u>Def.:</u> <u>1-dim. EFT</u>: functor $E: \mathcal{EB}^1 \to \mathcal{HS}$ compatible with the 3 add. structures.

② 1-dim. EFT's of degree n

Why do we need a degree structure?

1-dim. EFT's as model for K-theory

$\mathcal{EB}^1_n$: obj.: as in $\mathcal{EB}^1$

morph.: 1) pairs (f, c) with $f: Y_1 \to Y_2$ spin diffeom. and $c \in C(Y_1)^{-n}$

in part.: $f := (f, 1)$, $c := (id_{Y_1}, c)$

2) pairs (Σ, ψ) with $\Sigma: Y_1 \to Y_2$ 1-spin bordism with Riem. metric and $\psi \in \mathcal{F}(\Sigma)^{-n}$

in part.: $\Sigma := (\Sigma, \Omega^{-n})$ with Ω cyclic vacuum vector in $\mathcal{F}(\Sigma)$ if Σ has no closed components

$(\Sigma_1, \psi_1) = (\Sigma_2, \psi_2) \Leftrightarrow \Sigma_1 \cong \Sigma_2$ rel. bdry and $\mathcal{F}(\Sigma_1)^{-n} \xrightarrow{\cong} \mathcal{F}(\Sigma_2)^{-n}$

$\psi_1 \longmapsto \psi_2$

Comp. of morphisms ind. by comp. of diffeom., gluing of bordisms, mult. in Clifford alg., bimodul-op. on Fock spaces and gluing of Fock spaces.

<u>Remark:</u> All morphisms in $\mathcal{EB}^1_n$ are gen. by

1) $\varepsilon : pt \to pt$ and Clifford elements $c \in C(pt)^{-n} = C_{-n}$

2) $I_t : pt \to pt$

using comp., disj. union, inv. and adj. transf.

<u>in particular:</u>

$$(I_t, \psi) = (I_t, c_1 \Omega^{-n} c_2) = c_1 \circ (I_t, \Omega^{-n}) \circ c_2$$
$$= c_1 \circ I_t \circ c_2 \qquad , \; c_1, c_2 \in C_{-n}$$

<u>Def.:</u> <u>1-dim. EFT_n</u>: functor $E : \mathcal{EB}^1_n \to \mathcal{HS}$ comp. with the 3 add. structures and

<u>comp. with the linear str. on the morphisms:</u>

Given $f : Y_1 \to Y_2$ spin diffeom., $\Sigma : Y_1 \to Y_2$ 1-spin bordism, then

$$\begin{aligned} C(Y_1)^{-n} &\longrightarrow \mathcal{HS}(E(Y_1), E(Y_2)) \\ c &\longmapsto E(f, c) \end{aligned} \qquad \text{and}$$

$$\begin{aligned} \mathcal{F}(\Sigma)^{-n} &\longrightarrow \mathcal{HS}(E(Y_1), E(Y_2)) \\ \psi &\longmapsto E(\Sigma, \psi) \end{aligned}$$

are linear maps.

<u>Conclusions:</u> (Ex.)

1) $E(pt)$ graded left C_{-n} - module H with grading involution $E(\varepsilon)$ and Cl. mult. $E(C_{-n})$

2) $E(I_t): H \to H$ C_{-n} - linear

3) $E(I_t)$ even

4) $E(I_t)$ Hilbert - Schmidt

5) $E(I_t)$ self - adjoint

③ Standard example of 1-dim. EFT_n's

M Riem. spin mfd of dim. n, i.e. M comes with a graded irred. $C(T^*M)$ - C_n - bimodule bundle $Sp \to M$

$\to L^2(M, Sp)$ sep. Hilbert space & graded left C_{-n} - module

$\Rightarrow \{ \text{Op. } A : L^2(M, Sp) \to L^2(M, Sp) \}$ graded v.s.

Def. the Dirac operator

clifford mult.

$$D: C^\infty(M, Sp) \xrightarrow{\nabla} C^\infty(M, T^*M \otimes Sp) \xrightarrow{c} C^\infty(M, Sp)$$

$D: L^2(M, Sp) \longrightarrow L^2(M, Sp)$

- (unbounded) Fredholm op.
- commutes with the C_{-n} - op.
- odd
- self - adjoint

Let $t \in \mathbb{R}_{>0}$ $\rightsquigarrow$ heat op. for D^2: e^{-tD^2}

def. via functional calculus (ex.)

- Hilbert - Schmidt op.
- C_{-n} - linear
- even
- self - adjoint

Get a 1-dim. EFT_n E by setting

$E(pt) = L^2(M, Sp)$
$E(\varepsilon) =$ grad. inv. on $L^2(M, Sp)$
$E(c) =$ left mult. by c on $L^2(M, Sp)$
$E(I_t) = e^{-tD^2}$

④ 1-dim. susy EFT's (plain version)

Why do we need a super structure?

space of 1-dim. EFT_n's contractible (talk 10)

SEB^1 : obj.: (0|1)-super mfds with one-sided collars

$Y \hookrightarrow U(Y)$, i.e.

$$Y = \coprod_n \underset{\underset{\mathbb{R}^{0|1} \hookrightarrow [0,\varepsilon)^{1|1}}{\parallel}}{\mathrm{spt}} \; \coprod_m \underset{\underset{\overline{\mathbb{R}}^{0|1} \hookrightarrow (-\varepsilon,0]^{1|1}}{\parallel}}{\overline{\mathrm{spt}}}$$

with stand. metric   with stand. metric

morph.: 1) super diffeom. $f: Y_1 \to Y_2$, i.e.

f diffeom. of (0|1)-super mfds $Y_1 \to Y_2$ together with a diffeom. of the collars $U(Y_1) \to U(Y_2)$ that pres. the metric str.

2) 1-superbordisms $\Sigma: Y_1 \to Y_2$ with metric structure, i.e.

Σ (1|1)-super mfd with metric str. and $\partial\Sigma \cong \overline{Y}_1 \amalg Y_2$ (ex. bdry emb. $U(\overline{Y}_1) \hookrightarrow \Sigma \hookleftarrow U(Y_2)$ that pres. the metric str.)

$\Sigma_1 = \Sigma_2 \iff \Sigma_1 \underset{\substack{\text{metr.}\\\text{pres.}}}{\cong} \Sigma_2$ rel. bdry

Comp. of morphisms by gluing of bordisms:

$\Sigma_1 : Y_1 \to Y_2$, $\Sigma_2 : Y_2 \to Y_3$

$$U(\overline{Y}_1) \hookrightarrow \underset{\uparrow}{\Sigma_1} \cup_{Y_2} \underset{\uparrow}{\Sigma_2} \hookleftarrow U(Y_3)$$

$U(Y_2) \cup_{Y_2} U(\overline{Y}_2)$ = two-sided collar of Y_2

3 additional structures:

- symm. monoidal str.: disj. union
- involution $\overline{(\ )}$ and anti-involutions $(\ )^*$:
  $\overline{(\ )}$ = reversing the metric str. on collars and bordisms
  $(\ )^*$ = reversing the metric str. on bordisms
  (interpreted as reversal of direction)
- adj. transf.: $\mathcal{SEB}^1(\emptyset, \overline{Y}_1 \sqcup Y_2) \longrightarrow \mathcal{SEB}^1(Y_1, Y_2)$

Remark: (anal. to the non-susy case)
All morphisms in $\mathcal{SEB}^1$ are gen. by
1) kan. super involution ε on spt:
(on B-points) $\mathbb{R}^{0|1}(B) \longrightarrow \mathbb{R}^{0|1}(B)$ and
$\theta \longmapsto -\theta$

$$[0,\varepsilon)^{1|1}(B) \longrightarrow [0,\varepsilon)^{1|1}(B)$$
$$(t,\theta) \longmapsto (t,-\theta)$$

2) standard super intervals of length t,
I_t: spt $\to$ spt : Let $t \in \mathbb{R}_{>0}$.
$I_t = [0,t]^{1|1} = [0,t] \times \mathbb{R}^{0|1} \subseteq \mathbb{R}^{1|1}_{\geq 0}$ with stand. metric
and bdry emb. (on B-points)

$$[0,\varepsilon) \times \mathbb{R}^{0|1}(B) \hookrightarrow [0,t] \times \mathbb{R}^{0|1}(B) \hookleftarrow (-\varepsilon, 0]^{1|1} \times \mathbb{R}^{0|1}(B)$$
$$(t',\theta') \longmapsto (t',\theta')$$
$$(t'+t, \theta') \longleftarrow (t',\theta')$$

using comp., disj. union, inv. and adj. transf.

$\rightsquigarrow$ moduli space of 1-super bordisms $= \mathbb{R}_{>0}$

⑤ GSM-categories and GSM-functors

GSM = cat. of generalized super mfds:

<u>obj.</u>: functors $M: SM \longrightarrow Sets$ (gen. super mfds)

$$S \longmapsto M(S)$$

<u>morph.</u>: nat. transf. $T: M \longrightarrow N$

<u>Def.</u>: 1) A <u>GSM-category</u> (enriched cat.) $\mathcal{C}$ consists of

- a set of objects C
- for any two objects C_1, C_2 a gen. super mfd $\mathcal{C}(C_1, C_2)$ (hom-objects)
- for any 3 obj. C_1, C_2, C_3 a nat. transf. $\mathcal{C}(C_1, C_2) \times \mathcal{C}(C_2, C_3) \longrightarrow \mathcal{C}(C_1, C_3)$ (comp.)

2) The <u>underlying cat.</u> $\mathcal{UC}$ of $\mathcal{C}$ is given by evaluating the hom-objects on $\mathbb{R}^{0|0}$:

$$obj(\mathcal{UC}) = obj(\mathcal{C})$$
$$\mathcal{UC}(C_1, C_2) = \mathcal{C}(C_1, C_2)(\mathbb{R}^{0|0}) \quad \text{(hom-sets)}$$

3) A <u>GSM-functor</u> $\mathcal{F}: \mathcal{C} \longrightarrow \mathcal{D}$ between two GSM-cat. $\mathcal{C}, \mathcal{D}$ consists of

- a map on the objects $\mathcal{F}: obj(\mathcal{C}) \longrightarrow obj(\mathcal{D})$
- for any two objects C_1, C_2 of $\mathcal{C}$ a nat. transf. $\mathcal{F}(C_1, C_2): \mathcal{C}(C_1, C_2) \longrightarrow \mathcal{D}(\mathcal{F}(C_1), \mathcal{F}(C_2))$ resp. the composition

⑥ 1-dim. susy EFT's (enriched version)

Why do we need enrichment?

no super structure on the moduli space of 1-super bordisms

$\mathcal{SEB}^1_{en}$: obj.: as in $\mathcal{SEB}^1$

hom-objects: $S \mapsto$
1) families $f: Y_1 \to Y_2$ of super diffeom. parametrized by S
2) families $\Sigma: Y_1 \to Y_2$ of 1-super bordisms param. by S, i.e. $\Sigma \to S$ fiber bundle of metric 1-super bordisms

$\Sigma_1 = \Sigma_2 \iff \Sigma_1 \underset{\substack{\text{f.w.}\\ \text{metric}\\ \text{pres.}}}{\cong} \Sigma_2$ over S rel. bdry

Comp. of hom-objects by fiberwise gluing of bordisms

Thm: All hom-objects in $\mathfrak{SEB}^1_{en}$ are representable and therefore super mfds themselves.

Remark: All hom-objects in $\mathfrak{SEB}^1_{en}$ evaluated on S are gen. by

1) $\varepsilon : \mathrm{spt} \to \mathrm{spt}$ (as constant family)

2) super intervals param. by S, $I_{(t,\theta)} : \mathrm{spt} \to \mathrm{spt}$:
Let $(t,\theta) \in \mathbb{R}^{1|1}_{>0}(S)$.
Def.
$$\begin{array}{ccc} I_{(t,\theta)} & \subseteq & \mathbb{R}^{1|1}_{\geq 0} \times S \\ \downarrow & \swarrow_{\pi} & \\ S & & \end{array}$$
on $\mathbb{B}$-points:

Let $s \in S(\mathbb{B})$, then over s lie the emb.
$$\begin{array}{ccccc} & & \mathbb{R}^{1|1}_{\geq 0}(\mathbb{B}) & & \\ & & \cup\!\!\mid & & \\ [0,\varepsilon)^{1|1}(\mathbb{B}) & \xrightarrow{T_{(0,0)}} & I_{(t\circ s,\theta\circ s)} & \xleftarrow{T_{(t\circ s,\theta\circ s)}} & (-\varepsilon, 0]^{1|1}(\mathbb{B}) \\ (t',\theta') & \longmapsto & (t',\theta') & & \\ & & (t' + t\circ s + \theta'(\theta\circ s), \theta' + \theta\circ s) & \longleftarrow & (t',\theta') \end{array}$$

Note: param. by $S = \mathbb{R}^{0|0}$
$\rightsquigarrow I_{(t,\theta)} = I_t : \mathrm{spt} \to \mathrm{spt}$

using comp., disj. union, inv. and adj. transf.

$\rightsquigarrow$ moduli super mfd of 1-super bordisms $= \mathbb{R}^{1|1}_{>0}$

$\mathcal{HS}_{en}$: <u>obj.:</u> as in $\mathcal{HS}$

<u>hom-objects:</u> $S \longmapsto \mathcal{HS}(H_1, H_2) \otimes C^\infty(S)$

<u>Def.:</u> <u>1-dim. susy EFT</u>: GSM-functor

$$E: \mathcal{SEB}^1_{en} \longrightarrow \mathcal{HS}_{en}$$

comp. with the 3 add. structures.

<u>Remark:</u> underlying cat.:

$$\mathcal{U}\,\mathcal{SEB}^1_{en} = \mathcal{SEB}^1$$
$$\mathcal{U}\,\mathcal{HS}_{en} = \mathcal{HS}$$

$\rightsquigarrow \mathcal{U}E: \mathcal{SEB}^1 \longrightarrow \mathcal{HS}$ (plain version of 1-dim. susy EFT)

⑦ Standard example of 1-dim. susy EFT's

As in the non-susy case let $Sp \to M$ be a Riem. spin mfd and $D: L^2(M, Sp) \to L^2(M, Sp)$ be the corr. Dirac operator.

Let $(t, \theta) \in \mathbb{R}^{1|1}_{>0}(S)$.

$$\rightsquigarrow \underbrace{e^{-tD^2} + \theta D e^{-tD^2}}_{\text{def. via Taylor expansion calculus (ex.)}} \in \Big(\underbrace{HS}_{\text{Hilbert-Schmidt}}{}^{\overset{\text{self-adjoint}}{sa}}_{\underset{C_{-n}\text{-linear}}{C_{-n}}} (L^2(M, Sp)) \otimes C^\infty(S) \Big)^{\overset{\text{even}}{ev}}$$

Get a 1-dim. susy EFT E by setting

$E(spt) = L^2(M, Sp)$

$E(spt, spt)$ evaluated on S:

$E(\varepsilon) =$ grad. inv. on $L^2(M, Sp)$

$E(I_{(t,\theta)}) = e^{-tD^2} + \theta D e^{-tD^2}$

<u>Remark:</u> $\mathcal{U}E$ given by

$E(spt) = L^2(M, Sp)$

$E(\varepsilon) =$ grad. inv. on $L^2(M, Sp)$

$E(I_t) = e^{-tD^2}$

(analogy to the non-susy standard example)

⑧ Degree datum for susy EFT's

standard example: degree n - str. on the Hilbert side:
- $E(spt)$ left C_{-n} - module
- operators C_{-n} - linear

⇝ corresponding degree n-str. on the bordism side?

Idea: superstr. on mfds and bordisms
⟹ spin str. on the underlying mfds
⟹ def. of Clifford alg. and Fock-modules
⟶ degree datum

But: no connection of the degree str. to the original super str.
⟶ loss of information

⇝ "super degree datum" coming directly from the super structure?

WORK IN PROGRESS!

⇝ non-contractible spaces of field theories in each degree

⇝ spectrum for K-theory (talk 10)