Dirac Operators

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1 The connection form and the covariant derivative

We begin by briefly recalling some facts from the theory of connections.

Definition 1.1. Let (E, π, M) be a vector bundle, let M be a manifold. A *covariant derivative* is a linear map

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E),$$

which satisfies the Leibniz rule:

$$\nabla(f\Psi) = df \otimes \Psi + f\nabla\Psi$$

$\forall \in C^\infty(M)$ and for all $\Psi \in \Gamma(E)$.

Remark 1.2. If we take a vector field $X \in \Gamma(TM)$ and evaluate $\nabla_X\Psi$ at $x \in M$ then $(\nabla_X\Psi)(x)$ only depends on the vector X_x and the values of Ψ in an arbitrary small neigbourhood of x.

Definition 1.3. Let (P, π, M) be a G-principal bundle (G Lie group). For any point $p \in P$ there exists a canonical injection:

$$: \mathcal{G} \to T_pP$$
$$z \longmapsto \bar{z} = \frac{d}{dt}|_{t=0}(p\ exp(tz))$$

where $\mathcal{G}$ is the Lie algebra of G. Its image is called the *vertical space* V_p and is the tangent space to the fiber $\pi^{-1}(p)(i.eV_p = Ker(\pi_*))$.

Definition 1.4. Let (P, π, M) be a G-principal bundle. A connection on P is a G-invarian field of tangent n-planes (i.e $H_{pg} = (R_g)_*(H_p)$, where $R_g : P \to P$, $p \longmapsto pg$), such that:

$$T_pP = H_p \oplus V_p \text{ (}H_p \text{ horizontal subspace=complement of } V_p \text{ on } T_pP$$

The projection π induces an isomorphism

$$\pi_*|_{H_p} : H_p \to T_{\pi(p)}M.$$

Remark 1.5. G-invariant states that H_p and H_{pg} on the same fiber are related by R_{g*}.

2 The connection one-form

In practical computations, we need to separate T_pP into V_p and H_p in a systematic way. This can be achieved by introducing a Lie-algebra valued one form $\omega \in \mathcal{G} \otimes T^*P$ called the connection form.

Definition 2.1. A connection one-form $\omega \in \mathcal{G} \otimes T^*P$ is a projection of T_pP onto the vertical component V_p. The projection property is summarised by the following requirements:

1. $\omega_p(\bar{z}) = z, \quad \bar{z}$ as before
2. $R_g^*\omega = Ad(g^{-1}\omega$, i.e

$$\text{for all} X \in \Gamma(TP) \; \omega(R_g)_* X = Ad(g^{-1})\omega(X)$$

$$Ad : G \to End(\mathcal{G}), \quad g \longmapsto d\alpha_g, \text{ and } \alpha_g : G \to G, \; a \longmapsto gag^{-1}.$$

Define the horizontal subspace $H_p := Ker\omega_p$, then it defines a connection.

For a connection one-form ω on a G-principal fibre bundle (P, π, M), we define a covariant derivative on every associated vector bundle $E = P \times_\rho \Sigma$ as follows:

Take a section $\Psi \in \Gamma(E)$, which is locally given by $\Psi = [s, \sigma]$, where $s \in \Gamma_U(P)$ is a local section on $U \subset M$ and $\sigma : U \to \Sigma$ is a function. Since

$$TU \xrightarrow{s_*} TP \xrightarrow{\omega} \mathcal{G} \xrightarrow{\rho_*} End(\Sigma)$$

we can define a covarian derivative on E by:

$$\nabla_X \Psi := [s, X_\sigma + \rho_*((\omega \circ s_*)(X))\sigma]$$

for any $X \in TU$, where X_σ denotes the Lie derivative of σ in the direction of X.

3 The spinorial Levi-Civita connection

Notation

1. M denotes an n-dimensional Riemannian manifold with metric g.
2. $SOM = SO_n$-principal fibre bundle=natural fibre bundle of an oriented Riemannian manifold.
3. $(SpinM, \eta)$=spin structure on M.
4. $\Sigma M = SpinM \times_\rho \Sigma_n$ = Complex spinor bundle associated to a spin structure $SpinM$ of M.

Take a simply connected open subset $U \subset M$. Then any local section $s \in \Gamma_U(SOM)$ lifts to a section $\bar{s} \in \Gamma_U(SpinM)$, i.e,

$$\begin{array}{ccc} & & SpinM \\ & \overset{\bar{s}}{\nearrow} & \downarrow \\ U \subset M & \longrightarrow & SOM \end{array}$$

and we can define a connection one-form $\bar{\omega}$ on $SpinM$ as the unique connection one-form for which the following diagram commutes:

$$\begin{array}{ccccc} & & TSpinM & \xrightarrow{\bar{\omega}} & \mathfrak{spin}_n \\ & \overset{\bar{s}_*}{\nearrow} & \downarrow \eta_* & & \downarrow Ad_* \\ TU \subset TM & \xrightarrow[s_*]{} & TSOM & \xrightarrow[\omega]{} & \mathfrak{so}_n \end{array}$$

where $\mathfrak{spin}_n$ denotes the Lie algebra of $Spin_n$ and $\mathfrak{so}_m$ denotes the Lie algebra of SO_n, which is the space of real skew-symmetric matrices. Hence a one-form can be considered as an $n \times n$ matrix of one-forms $\omega = ((\omega_{ij}))$, $\omega_{ij} = -\omega_{ji}$.

To get a local description of the associated covarian derivative ∇ on ΣM, take an orthonormal frame $s = (e_1, \ldots, e_n) \in \Gamma_U(SOM)$ $U \subset M$, and denote by:

$$\omega := s^*\omega = -\sum_{i<j} \omega_{ij} e_i \wedge e_j,$$

where $e_i \wedge e_j := g(e_i, .)e_j - g(e_j, .)e_i$ is a basis of $\mathfrak{so}_n$. We then get

$$\omega_{ij}(X) = -g(\omega(X)e_i, e_j) = -g(\nabla_X e_i, e_j)$$

for all $X \in \Gamma(TM)$.

4 Dirac Operator

In the following we wil often use a local orthonormal frame denoted by $s = (e_1, \ldots, e_n) \in \Gamma_u(SOM)$, $U \subset M$, which yields the relation

$$e_i.e_j + e_j.e_i = -2\delta_{ij} \; 1 \leq i, j \leq n$$

In talk number 4, we have seen that associated to a spin structure of a Riemannian manifold (M^n, g), there are three essential structures:

1. The spinor bundle $\Sigma M = SpinM \otimes_\rho \Sigma_n$, with the Clifford multiplication

$$m: \quad TM \otimes \Sigma M \longrightarrow \Sigma M$$

$$X \otimes \Psi \longmapsto X.\Psi := \rho(X)\Psi,$$

where ρ is the spinor representation. This multiplication extends to

$$m: \quad \bigwedge^p(TM) \otimes \Sigma M \longrightarrow \Sigma M$$

$$\alpha \quad \otimes \quad \Psi \quad \longmapsto \quad \textstyle\sum_{1\leq i_1<\cdots<i_p\leq n} \alpha_{i_1\ldots i_p} e_{i_1} \ldots e_{i_p}.\Psi$$

where locally

$$\alpha = \sum_{1\leq i_1<\cdots<i_p\leq n} e^*_{i_1} \wedge \cdots \wedge e^*_{i_p}$$

and $e^*_i = g(e_i, .)$ is the dual basis of e_i.

2. The natural Hermitian product $(.,.)$ on sections of ΣM.
3. The Levi-Civita connection on ΣM.

Moreover, these structures satisfy the following compatibility conditions:

1. $(X.\Psi, \phi) + (\Psi, X.\phi) = 0$
2. $X(\Psi, \phi) - (\nabla_X \Psi, \phi) - (\Psi, \nabla_{X\phi}) = 0$
3. $\nabla_X(Y.\Psi) - \nabla_X Y.\Psi - Y.\nabla_X \Psi = 0$

for all $X, Y \in \Gamma(TM)$, $\Psi, \phi \in \Gamma(\Sigma M)$.

Definition 4.1. The *Dirac operator* is the composition of tha covariant derivative acting on sections of ΣM with the Clifford multiplication

$$D := m \circ \nabla.$$

Locally, we get:

$$D: \quad \Gamma(\Sigma M) \xrightarrow{\nabla} \Gamma(T^*(M \otimes \Sigma M) \xrightarrow{m} \Gamma(\Sigma M)$$

$$\Psi \longmapsto \textstyle\sum_{i=1}^n e^*_i \otimes \nabla_{e_i}\Psi \longmapsto \sum_{i=1}^n e_i.\nabla_{e_i}\Psi$$

Lemma 4.2. The commutator of the Dirac operator with the action, by pointwise multiplication on the spinor bundle, of a function $f : M \to \mathbb{C}$, is given by:

$$[D, f]\Psi := df.\Psi, \quad \Psi \in \Gamma(\Sigma M)$$

Proof A locally calculation shows that:

$$\begin{array}{rcl} [D, f]\Psi & = & (Df - fD)\Psi = \sum_{i=1}^{n} e_i.\nabla_{e_i}(f\Psi) - fD\Psi \\ & = & \sum_{i=1}^{n} df(e_i)e_i.\Psi + fD\Psi - fD\Psi \\ & = & df.\Psi \end{array}$$

Lemma 4.3. The Dirac operator is a first order partial differential operator which is

1. elliptic (i.e for all $\xi \in T^*M - \{0\}$, $\sigma_\xi(D) : \Sigma_x M \to \Sigma_x M$, $\sigma_\xi(D)(\Psi(x)) := \xi.\Psi(x)$ (Clifford multiplication by ξ.) is an isomorphism. ($\sigma(D)$ is called *principal symbol.*

2. and formally self-adjoint with respect to

$$(.,.)_{L^2} := \int_M (.,.)\nu_g,$$

if M is compact, and where ν_g denotes the volumen element.

sketch of the proof:

1. $\sigma_\xi(D) : \Sigma_x M \to \Sigma_x M$ is an isomorphism $\xi.\Psi = 0 \longrightarrow \xi.\xi.\Psi = 0 \longleftrightarrow -||\xi||^2\Psi = 0 \longleftrightarrow \Psi = 0$

2. To show D is self-adjoint choose normal coordinates at $x \in M$ i.e $(\nabla_{e_i} e_j)(x) = 0$ $1 \leq i, j, \leq n$, and compute $(D\Psi, \varphi)$.Now, use the following :

$$X(\Psi, \varphi) - (\nabla_X \Psi, \varphi) - (\Psi, \nabla_X \varphi) = 0$$

to show that:

$$(D\Psi, \varphi) = |_x - \sum_{i=1}^{n} e_i(\Psi, e_i.\varphi) + (\Psi, D\varphi)$$

Finally prove that: $(D\psi, \varphi) = -divX_1 - idivX_2 + (\Psi, D\varphi)$, this last equation does not depend on the choice of coordinates, so

$$\int_m (D\Psi, \varphi)\nu_g = \int_M (\varphi, D\Psi)\nu_g,$$

since $\partial M = \emptyset$.

Lemma 4.4. For $n = 2m$

1.
$$D : \Gamma(\Sigma^{\pm} M) \to \Gamma(\Sigma^{\mp} M),$$
i.e the Dirac operator sends positive spinors into negative spinors.

2. The eigenvalues of D are symmetric with respect to the origin.

Examples: *Dirac Operator*

1. Let $M = \mathbb{R}^n$, $\Sigma\mathbb{R}^n = \mathbb{R}^n \times \mathbb{C}^N$, with $N = 2^{[\frac{n}{2}]}$. This implies that every spinor $\Psi \in \Gamma(\Sigma\mathbb{R}^n)$ is a function $\Psi : \mathbb{R}^n \to \mathbb{C}^N$. The, the Dirac operator is given by:
$$D = \sum_{i=1}^{n} e_i.\partial_i$$
which acts on differential maps from $\mathbb{R}^n$ to $\mathbb{C}^n$, where $\partial_i = \nabla_{e_i}$.

2. Let $n = 2$, and $M = \mathbb{R}^2$. Let $\mathbb{C}l_2$ be the complexification of the Clifford real algebra Cl_n, which is isomorphic to the group of 2×2 matrices. Then $\Sigma_2 = \Sigma_2^+ \otimes \Sigma_2^- = \mathbb{C} \oplus \mathbb{C}$, where $\Sigma_2^+ = span_{\mathbb{C}}(e_1 + ie_2)$ and $\Sigma_2^- = span_{\mathbb{C}}(1 - e_1.e_2)$. Then $\Psi \in \Gamma(\Sigma M)$ is given by complex functions
$$\Psi = f(e_1 + ie_2) + g(1 - ie_1.e_2)$$
The Dirac operator is given by:
$$\begin{aligned} D\Psi &= (e_1.\partial_1 + e_2.\partial_2)[(e_1 + ie_2)f + (1 - ie_1.e_2)g] \\ &= -(\partial_1 + i\partial_2)f(1 - ie_1.e_2) + (\partial_1 - i\partial_2)g(e_1 + ie_2) \\ &= 2(-\partial_{\bar{z}} f(1 - e_1.e_2) + \partial_z g(e_1 + ie_2)), \end{aligned}$$
where $\partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2)$ and $\partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$. That is
$$\begin{pmatrix} 0 & 2\partial_z \\ -2\partial_{\bar{z}} & 0 \end{pmatrix}$$
in the basis $\{(e_1 + ie_2), (1 - ie_1.e_2)\}$ of Σ_2. Hence the Dirac operator D can be considered as a generalization of the Cauchy Riemann operator.

5 Spin structures on conformal manifolds

Let Σ be a d-dimensional manifold, let $k \in \mathbb{R}$. Let $L^k \to \Sigma$ be an oriented real line bundle which fiber over $x \in \Sigma$ consists of all maps $\rho : \bigwedge^d(T_x\Sigma) \to \mathbb{R}$, such that, $\rho(\lambda\omega) = (|\lambda|^{\frac{k}{d}}\rho(\omega))$ for all $\lambda \in \mathbb{R}$. Sections of L^d are refered to as densities (weights). They can be integrated over Σ resulting in a real number.

From now, Σ is assumed to be equipped with a conformal strcuture (i.e an equivalence class of Riemannian metrics, where we identify a metric obtained by multiplication by a function with the original metric).

Remark 5.1. For any $k \neq 0$ the choice of a metric in the conformal class corresponds to the choice of a positive section L^k. Moreover, the conformal structure on Σ induces a canonical Riemannian metric on the *weightless cotangent bundle* $T_0^*\Sigma := L^{-1} \otimes T^*\Sigma$.

The metric on $T_0^*\Sigma$ is defined as follows: Let $\sigma \in \Gamma(\Sigma, T^*\Sigma)$ and let $\rho \in \Gamma(\Sigma, L^{-1})$. Then $\sigma \otimes \rho \in \Gamma(T_0^*\Sigma)$, hence we define a metric on $T_0^*\Sigma$ as:

$$||\sigma \otimes \rho||_{[g]} := \rho(Vol_g).||\sigma||_g.$$

It is well defined for a conformal class, because:

$$\begin{array}{rcl} \text{If } g' = fg \text{ then} & & \\ \rho(Vol_{g'}).||\sigma||_{g'} & = & \frac{1}{(||f||)^{\frac{1}{2}}}\rho(Vol_g).(||f||)^{\frac{1}{2}}||\sigma||_g \\ & = & \rho(Vol_g).||\sigma||_g \end{array}$$

Definition 5.2. A *spin structure* on a conformal d-manifold Σ is by definition a spin structure on the Riemannian vector bundle $T_0^*\Sigma$.

Let Σ^d be a conformal spin manifold. Picking a Riemannian metric in the conformal class determines the Levi-Civita connection on the tangent bundle of Σ, which in turn determines connections on the spinor bundle $S = S(T_0^*\Sigma)$, the line bundles L^k and hence $L^k \otimes S$ for all $k \in \mathbb{R}$.

Definition 5.3. The *Dirac operator* on weighted spinor bundle $D = D_\Sigma$ is the composition:

$$D : C^\infty(\Sigma; L^k \otimes S) \xrightarrow{\nabla} C^\infty(\Sigma; T^*\Sigma \otimes L^k \otimes S) = C^\infty(\Sigma; L^{k+1} \otimes T_0^*\Sigma \otimes S)$$

$$\xrightarrow{c} C^\infty(\Sigma; L^{k+1} \otimes S)$$

where c is the Clifford multiplication (given by the left action of $T_0^*\Sigma \subset c(T_0^*\Sigma)$ on S.) ∇ is the connection on $L^k \otimes S$.

Remark 5.4. For $k = \frac{d-1}{2}$, the Dirac operator is independent of the choice of the Riemannian metric. See [1]

Let Σ^d be a conformal spin manifold with boundary Y. Assume that the bundle ξ extends to a vector bundle with metric an connection on Σ. We denote it again by ξ and let $\partial\xi$ its restriction to Y. Let S be the spinor bundle of Σ an recall that the restiction of S^+ to Y is the spinor bundle of Y.

Definition 5.5. The *twisted Dirac operator* is the composition:

$$D_\xi : C^\infty(\Sigma; L^{\frac{d-1}{2}} \otimes S \otimes \xi) \xrightarrow{\nabla} C^\infty(\Sigma; T^*\Sigma \otimes L^{\frac{d-1}{2}} \otimes S \otimes \xi)$$

$$= C^\infty(\Sigma; L^{\frac{d+1}{2}} \otimes T_0^*\Sigma \otimes S \otimes \xi)$$

$$\xrightarrow{c} C^\infty(\Sigma; L^{\frac{d+1}{2}} \otimes S \otimes \xi)$$

where ∇ is the connection on $L^{\frac{d-1}{2}} \otimes S \otimes \xi$ determined by the connection on ξ and the Levi-Civita connection on $L^{\frac{d-1}{2}} \otimes S$ for the choice of a metric given in the conformal class.

6 Index of Dirac operator

Fact: Over a compact manifold, the kernel and cokernel of an elliptic operator P are of finite dimension.

Definition 6.1. The *index* of P is definided as:

$$indP := dim(kerP) - dim(cokerP)$$

Example: Let X be a compact Riemannian manifold of dimension $4m$. Consider the complex spinor bundle $\not{\mathbb{S}}_{\mathbb{C}}$, with Dirac operator $\not{\mathbb{D}}$. We split $\not{\mathbb{S}}_{\mathbb{C}} \cong \not{\mathbb{S}}_{\mathbb{C}}^+ \oplus \not{\mathbb{S}}_{\mathbb{C}}^-$, where $\not{\mathbb{S}}_{\mathbb{C}}^\pm = (1 \pm \omega_{\mathbb{C}}) \not{\mathbb{S}}_{\mathbb{C}}$, with $\omega_{\mathbb{C}}$ the complex volume elelement, given in terms of a positive oriented tangent frame $(e_1, \ldots, e_{2m})$.

$$\omega_{\mathbb{C}} = i^m e_1 \ldots e_{2m}$$

This is a globally defined section of

$$\mathbb{C}l(C) = Cl(X) \otimes \mathbb{C},$$

with properties:

1. $\nabla_{\omega_{\mathbb{C}}} = 0$
2. $\omega_{\mathbb{C}}^2 = 1$
3. $\omega_{\mathbb{C}} e = -e\omega_{\mathbb{C}}$, for any $e \in TX$.

Theorem 6.2. Let X be a compact spin manifold of dimension $2m$. Consider

$$\not{D}^+ : \Gamma(\mathbb{S}_{\mathbb{C}}^+(X)) \to \Gamma(\not{\mathbb{S}}_{\mathbb{C}}^-(\mathbb{C}))$$

Then

$$ind\ \not{D}^+ = \hat{A}(X).$$

More general: If E is any complex vector bundle over X, then index of

$$\not{D}_E^+ : \Gamma(\not{\mathbb{S}}_{\mathbb{C}}^+(X) \otimes E) \to \Gamma(\not{\mathbb{S}}_{\mathbb{C}}^-(X) \otimes E)$$

is

$$ind(\not{D}_E^-) = (chE.\hat{\mathbf{A}})[X]$$

Theorem 6.3. Let X be a compact oriented manifold of dimension $2m$. Consider

$$D^+ : \Gamma(\mathbb{C}l^+(X)) \to \Gamma(\mathbb{C}l^-(X))$$

Then

$$indD^+ = L(X) = sig(X)$$

In general, if E is any complex vector bundle over X, then

$$D_E^+ : \Gamma(\mathbb{C}l^+(X) \otimes E) \to \Gamma(\mathbb{C}l^-(X) \otimes E)$$

is given by:

$$ind(D_E^+) = (ch_2E.L(X))[X]$$

where $ch_2E = \sum_k 2^k ch^k E$, and $ch^k E = \frac{1}{k!}\sum_{i=1}^n x_i^n$.

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