Exercises in Geometry II
University of Bonn, Summer Semester 2018
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Sheet 10

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1. Klingenberg's Lemma [8 points]

Let (M, g) be a complete Riemannian manifold with sectional curvature $\sec \leq C$, where C is a positive constant. Let $p, q \in M$ and let γ_0 and γ_1 be two distinct geodesics joining p to q with $L(\gamma_0) \leq L(\gamma_1)$. Assume that γ_0 is homotopic to γ_1, i.e. there exists a continuous family of curves $(\alpha_t)_{t\in[0,1]}$ joining p to q such that $\alpha_0 = \gamma_0$ and $\alpha_1 = \gamma_1$.
The aim of this exercise is to show that there exists a $t_0 \in [0, 1]$ such that

$$L(\gamma_0) + L(\alpha_{t_0}) \geq \frac{2\pi}{\sqrt{C}}.$$

a) Without loss of generality, we can assume that $L(\gamma_0) < \frac{\pi}{\sqrt{C}}$ (Why?). Consider the exponential map $\exp_p : T_pM \to M$ and let B be the ball of radius $\frac{\pi}{\sqrt{C}}$ around $0 \in T_pM$.

   Show that for small t the curve α_t can be lifted to a curve $\tilde{\alpha}_t$ in T_pM joining $\exp_p^{-1}(p) = 0$ and $\exp_p^{-1}(q) = \tilde{q}$ such that $\exp_p \circ \tilde{\alpha}_t = \alpha_t$.

b) Show that for all small $\varepsilon > 0$ there exists a $t(\varepsilon)$ such that $\alpha_{t(\varepsilon)}$ can be lifted to a curve $\tilde{\alpha}_{t(\varepsilon)}$ that contains points with distance $< \varepsilon$ from the boundary ∂B of B.

c) Show that $T := \{t \in [0, 1] : \alpha_t \text{ can be lifted to } T_pM\}$ is a strict subset of $[0, 1]$, i.e. $T \neq [0, 1]$.

d) Conclude Klingenberg's Lemma.

2. A new proof of the Hadamard Theorem [4 points]

Use Klingenberg's Lemma from the last exercise for the proof of Hadamard's Theorem: Let (M, g) be an n-dimensional complete Riemannian manifold, simply connected with sectional curvature $\sec \leq 0$. Then M is diffeomorphic to $\mathbb{R}^n$. More precisely, $\exp_p : T_pM \to M$ is a diffeomorphism.

Hint: Take $C = 1/n$, for an integer n, in Klingenberg's lemma and show that if M is simply connected, then there exists a unique geodesic joining the points $p, q \in M$.

3. The Sturm comparison theorem [4 points]

In this exercise we will do a direct proof of Rauch's Theorem in dimension two using the Sturm Comparison Theorem.
Let

$$f''(t) + K(t)f(t) = 0, \ f(0) = 0,$$
$$\tilde{f}''(t) + \tilde{K}(t)\tilde{f}(t) = 0, \ \tilde{f}(0) = 0,$$

with $t \in [0, l]$, be two ordinary differential equations. Suppose that $\tilde{K}(t) \geq K(t)$ for all t, and that $f'(0) = \tilde{f}'(0) = 1$.

a) Show that for all $t \in [0, l]$,

$$0 = \int_0^t \left(\tilde{f}(f'' + Kf) - f(\tilde{f}'' + \tilde{K}\tilde{f}) \right) dt \tag{1}$$

$$= \left[\tilde{f}f' - f\tilde{f}' \right]_0^t + \int_0^t (K - \tilde{K}) f \tilde{f} dt \tag{2}$$

and conclude from this identity that the first zero of f does not occur before the first zero of $\tilde{f}$, i.e. if $\tilde{f}(t) > 0$ on $(0, t_0)$ and $\tilde{f}(t_0) = 0$, then $f(t) > 0$ on $(0, t_0)$.

b) Suppose that $\tilde{f}(t) > 0$. Use part a) to show that $f(t) \geq \tilde{f}(t), \ t \in [0, l]$, and that the equality is verified for $t = t_1 \in (0, l]$ if and only if $K(t) = \tilde{K}(t), \ t \in [0, t_1]$.

c) Conclude that part b) proves Rauch's Theorem in dimension two.

Due on Monday, July 16.

Homepage of the lecture: `https://www.math.uni-bonn.de/people/galazg/`