The Universality Classes in the Parabolic Anderson Model

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The Parabolic Anderson Model

We consider the Cauchy problem for the heat equation with random coefficients and localised initial datum:

$$\frac{\partial}{\partial t} u(t,z) = \Delta^{\mathrm{d}} u(t,z) + \xi(z) u(t,z), \qquad \text{for } (t,z) \in (0,\infty) \times \mathbb{Z}^d, \tag{1}$$

$$u(0,z) = \mathbb{1}_0(z), \qquad \text{for } z \in \mathbb{Z}^d. \tag{2}$$

- $\xi = (\xi(z) \colon z \in \mathbb{Z}^d)$ i.i.d. random potential, $[-\infty, \infty)$-valued.
- $\Delta^{\mathrm{d}} f(z) = \sum_{y \sim z} \big[f(y) - f(z) \big]$ discrete Laplacian
- $\Delta^{\mathrm{d}} + \xi$ Anderson Hamiltonian

The solution $u(t, \cdot)$ is a random time-dependent shift-invariant field. Its a.s. existence is guaranteed under a mild moment condition on the potential. It has all moments finite if all positive exponential moments of $\xi(0)$ are finite.

Motivations and Feynman-Kac formula

Interpretations / Motivations:

- Random mass transport through a random field of sinks and sources.
- Expected particle number in a branching random walk model in a field of random branching and killing rates.
- Linearised model for chemical kinetics, equivalent to Burger's equation in hydrodynamics, describes magnetic phenomena.

Background literature and surveys: [MOLCHANOV 1994], [CARMONA/MOLCHANOV 1994], [SZNITMAN 1998], [GÄRTNER/K. 2005].

Main tool for analysis: Feynman-Kac formula

$$u(t,z) = \mathbb{E}_z\Big[\exp\Big\{\int_0^t \xi(X(s))\,\mathrm{d}s\Big\}\mathbb{1}\{X(t)=0\}\Big], \qquad z\in\mathbb{Z}^d, t>0,$$

where $(X(s))_{s\in[0,\infty)}$ is a simple random walk on $\mathbb{Z}^d$ with generator Δ^{d}, starting from z under $\mathbb{P}_z$.

Questions and Heuristics I

MAIN GOAL: Describe the large-t behavior of the solution $u(t,\cdot)$. In particular: where does the main bulk of the total mass stem from?

Total mass of the solution:

$$U(t) = \sum_{z\in\mathbb{Z}^d} u(t,z), \qquad \text{for } t>0.$$

Much work is devoted to a thorough understanding of the effect of

Intermittency: Asymptotically as $t\to\infty$, the main contribution to $U(t)$ comes from few small remote islands.

- These islands are randomly located, t-dependent, not too far from the origin.
- Both the solution $u(t,\cdot)$ and the potential $\xi(\cdot)$ are exceptionally large in these islands.
- The large-t behavior is determined by the largest eigenvalue of the Anderson Hamiltonian $\Delta^{\mathrm{d}}+\xi$ (i.e., by the bottom of the spectrum of $-\Delta^{\mathrm{d}}-\xi$) in large t-dependent boxes.
- This in turn is determined by the extreme values of the potential ξ.
- Hence, only the upper tails of $\xi(0)$ matter.

Questions and Heuristics II

If all the moments $\langle U(t)^p \rangle$ of $U(t)$ are finite for any $p, t > 0$, then intermittency can be characterised by the requirement

$$0 < p < q \qquad \Longrightarrow \qquad \limsup_{t \to \infty} \frac{\langle U(t)^p \rangle^{1/p}}{\langle U(t)^q \rangle^{1/q}} = 0.$$

Presence of intermittency [Gärtner/Molchanov 1990]: In this sense, intermittency holds as soon as the potential is not a.s. constant.

Deeper questions:

(1) How large are the islands, and how large are the potential and the solution there?

(2) What do the shapes of the potential and the solution look like in the islands?

(3) How many islands are there?

Summary of Answers

- (1) and (2) have been answered in a strong sense in two main special cases ([Sznitman 1998], [Gärtner/Molchanov/König 2006]). That is, a proof was given that the complement of certain islands is negligible.
- (1) and (2) have been answered in a less rigorous sense in basically all cases in which the positive exponential moments of the potential are finite ([Gärtner/Molchanov 1998], [Biskup/K. 2001], [van der Hofstad/K./Mörters 2005]). That is, the moment asymptotics and the almsot sure asymptotics of $U(t)$ were identified.
- Under some mild regularity assumption, the case of finite positive exponential moments has been completely classified in four universality classes ([van der Hofstad/K./Mörters 2005]).
- (3) is open for all the classes of finite positive exponential moments, but has recently been answered for heavy-tailed potentials ([K./Mörters/Sidorova 2006]).

The double-exponential distribution I

We consider the potential distribution determined by the upper tails

$$\mathrm{Prob}(\xi(0) > r) \approx \exp\Big\{ - \mathrm{e}^{r/\varrho} \Big\}, \qquad r \to \infty,$$

where $\varrho \in (0, \infty]$ is a parameter. Then the cumulant generating function, $H(t) = \log\langle \mathrm{e}^{t\xi(0)} \rangle$, turns out to be $H(t) = \rho t \log(\rho t) - \rho t + o(t)$.

Moment asymptotics. [GÄRTNER/MOLCHANOV 1998]: For any $p \in \mathbb{N}$, as $t \to \infty$,

$$\langle U(t)^p \rangle = \mathrm{e}^{H(tp)} \mathrm{e}^{-tp(\chi + o(1))}, \qquad \text{where} \qquad \chi = \inf_{\varphi\colon \mathbb{Z}^d \to \mathbb{R}} \Big[\mathcal{L}(\varphi) - \lambda(\varphi) \Big],$$

and $\mathcal{L}(\varphi) = \frac{\varrho}{\mathrm{e}} \sum_{z \in \mathbb{Z}^d} \mathrm{e}^{\varphi(z)/\varrho}$, and $\lambda(\varphi)$ is the top of the spectrum of $\Delta^{\mathrm{d}} + \varphi$ in $\mathbb{Z}^d$.

- Minimiser(s) exist. They are unique for ϱ sufficiently large and rather inexplicit.
- $\mathcal{L}(\varphi)$ is a large-deviation rate function for the shifted potential $\xi_t = \xi - H(t)/t$. On the event $\{\xi_t \approx \varphi\}$, the contribution to the Feynman-Kac formula is quantified by $\lambda(\varphi)$. The optimal profile describes the total expected mass.
- The structure of the asymptotic optimal profile is discrete; no spatial scaling is involved.

The double-exponential distribution II

Almost sure asymptotics. [GÄRTNER/MOLCHANOV 1998]: As $t \to \infty$,

$$\frac{1}{t}\log U(t) = \frac{H(\log t)}{\log t} - \widetilde{\chi} + o(1), \qquad \text{where} \qquad -\widetilde{\chi} = \sup\Big\{\lambda(\varphi)\Big|\varphi\colon \mathbb{Z}^d \to \mathbb{R}, \mathcal{L}(\varphi) \leq d\Big\}.$$

- The maximiser(s) of χ and $\widetilde{\chi}$ are identical.
- A Borel-Cantelli argument shows that, with probability one, for all sufficiently large t, every potential shape φ satisfying $\mathcal{L}(\varphi) \leq d$ appears on some island in the box $[-t,t]^d \cap \mathbb{Z}^d$ in the potential $\xi_{\log t} = \xi - H(\log t)/\log t$. The contribution to the Feynman-Kac formula coming from those paths that go quickly there and spend most of the time there is quantified by $\lambda(\varphi)$. The optimal such profile φ describes the total contribution.
- Every such island with φ an optimal profile is potentially one of the intermittent islands we mentioned above.

The double-exponential distribution III

Geometric characterisation of intermittency. [GÄRTNER/K./MOLCHANOV 2006]: Almost surely, for any sufficiently large t, there is a random set $\Gamma_t \subset [-t,t]^d \cap \mathbb{Z}^d$ such that $|\Gamma_t| = t^{o(1)}$ and

- $\min_{z,\widetilde{z}\in\Gamma_t\colon z\neq\widetilde{z}} |z-\widetilde{z}| = t^{1-o(1)}$,
- $\lim_{R\to\infty} \liminf_{t\to\infty} \frac{1}{U(t)} \sum_{z\in B_R(\Gamma_t)} u(t,z) = 1$,
- For any $z\in\Gamma_t$ and any $R>0$, in the box $B_R(z)$, the shifted potential $\xi_{\log t}$ resembles the maximiser φ of $-\widetilde{\chi}$, and the solution $u(t,\cdot)$ resembles C_t times the principal eigenfunction of $\Delta^{\mathrm{d}} + \varphi$ for some suitable norming $C_t > 0$.

- Main tools of the proof: probabilistic cluster expansion, Borel-Cantelli arguments, deeper analysis of the variational formula $\widetilde{\chi}$.
- This geometric picture of intermittency is in accordance with Anderson localisation theory. Indeed, (the upper part of) the spectrum of the Anderson Hamiltonian $\Delta^{\mathrm{d}} + \xi$ is pure point, and we may expand

$$u(t,\cdot) = \sum_k \mathrm{e}^{\lambda_k t}\, \mathrm{e}_k(0)\mathrm{e}_k(\cdot).$$

The eigenfunctions e_k are expected to be exponentially localised and sparsely distributed in space.

Potentials bounded from above I

We assume that $\text{esssup}(\xi(0)) = 0$. With a parameter $\gamma \in [0, 1)$, we consider potentials having the upper-tail behavior

$$\text{Prob}(\xi(0) > -x) \approx \exp\Big\{ -\text{const.}\, x^{-\gamma/(1-\gamma)} \Big\}, \qquad x \downarrow 0.$$

- The case $\gamma = 0$ contains the case of i.i.d. Bernoulli traps, where $\xi(0) \in \{-\infty, 0\}$.
- $H(t) = \log\langle \mathrm{e}^{t\xi(0)} \rangle$ is roughly $H(t) \approx -\text{const.}\, t^{\gamma}$.

It turns out that the moments are determined by islands whose diameter is of the order

$$\alpha(t) = t^{\nu}, \qquad \text{where} \qquad \nu = \frac{1-\gamma}{d+2-d\gamma} \in \big(0, \tfrac{1}{d+2}\big].$$

Moment asymptotics. [BISKUP/K. 2001]: For any $p \in (0, \infty)$, as $t \to \infty$,

$$\frac{1}{tp} \log\langle U(t)^p \rangle = -\frac{\chi + o(1)}{\alpha(pt)^2}, \qquad \text{where} \qquad \chi = \inf_{\varphi \in \mathcal{C}(\mathbb{R}^d \to [-\infty, 0])} \Big[\mathcal{L}(\varphi) - \lambda(\varphi) \Big],$$

and $\mathcal{L}(\varphi) = \text{const.} \int_{\mathbb{R}^d} |\varphi(x)|^{-\gamma/(1-\gamma)}\, \mathrm{d}x$ for $\gamma \in (0, 1)$ and $\mathcal{L}(\varphi) = \text{const.}\, |\text{supp}(\varphi)|$ for $\gamma = 0$, and $\lambda(\varphi)$ is the top of the spectrum of $\Delta + \varphi$ in $L^2(\mathbb{R}^d)$.

Potentials bounded from above II

- $\mathcal{L}$ is a large-deviation rate function for the rescaled potential $\xi_t(\cdot) = \alpha(t)^2 \xi(\cdot\, \alpha(t))$.
- For $\gamma = 0$, the formula χ is well-analysed. The minimiser is unique up to shifts and has compact support and can be characterised in terms of Bessel functions.
- The special case of Bernoulli traps is the discrete analogue of Brownian motion among Poisson obstacles, see [SZNITMAN 1998], [ANTAL 1994, 1995], [BOLTHAUSEN 1994], [POVEL 1999] and others for finer investigations.

Almost sure asymptotics. [BISKUP/K. 2001]: Almost surely, as $t \to \infty$,

$$\frac{1}{t} \log U(t) = -\frac{\widetilde{\chi} + o(1)}{\alpha(\beta(t))^2}, \qquad \text{where} \qquad -\widetilde{\chi} = \sup\Big\{\lambda(\varphi)\colon \varphi \in \mathcal{C}_-(\mathbb{R}^d), \mathcal{L}(\varphi) \leq d\Big\},$$

where $\beta(t)$ is defined by $\beta(t)/\alpha(\beta(t))^2 = d \log t$.

- Interpretation and proof are analogous to the double-exponential case.
- The variational formulas χ and $\widetilde{\chi}$ have the same minimiser(s) if any.
- Deeper investigations concerning the geometric description of intermittency have been done yet only for the special case of Brownian motion among Poisson obstacles, the continuous version, see [SZNITMAN 1998] (however, not from the view point of the parabolic Anderson model).

The universality classes

Intuitively, both $\varrho \approx 0$ in the double-exponential case and $\gamma \approx 1$ in the bounded case refer to 'almost bounded', coming from the two sides. Is there an interesting class in the intersection of the cases '$\varrho = 0$' and '$\gamma = 1$'? Are these then all the interesting classes?

The Universality Classes. [VAN DER HOFSTAD/K./MÖRTERS 2005]). Assume that

$$\lim_{t\to\infty} \frac{H(ty) - yH(t)}{\kappa(t)} = \widehat{H}(y), \qquad y > 0,$$

for some continuous scale function κ and some nontrivial function $\widehat{H}\colon (0,\infty) \to \mathbb{R}$.
Furthermore, assume that $\kappa^* = \lim_{t\to\infty} \kappa(t)/t \in [0,\infty]$ exists.
Then there are $\gamma \in [0,\infty)$ and $\varrho \in (0,\infty)$ such that

(i) κ is regularly varying with index γ (in particular, $\kappa(t) = t^{\gamma+o(1)}$),

(ii) if $\gamma \neq 1$, then $\widehat{H}(y) = \text{const.}\,\frac{y-y^\gamma}{1-\gamma}$, and if $\gamma = 1$, then $\widehat{H}(y) = \varrho y \log y$.

Then there are precisely four cases:

(1) $\gamma > 1$, or $\gamma = 1$ and $\kappa^* = \infty$: double-exponential case with $\varrho = \infty$.

(2) $\gamma = 1$ and $\kappa^* \in (0,\infty)$: double-exponential case with $\varrho \in (0,\infty)$.

(3) $\gamma = 1$ and $\kappa^* = 0$: almost-bounded case (see below).

(4) $\gamma < 1$: bounded case.

Almost-bounded potentials

The moment asymptotics in all the four universality classes are given by

$$\frac{1}{tp}\log\langle U(t)^p\rangle = \frac{H\big(pt\alpha(pt)^{-d}\big)}{pt\alpha(pt)^{-d}} - \frac{\chi + o(1)}{\alpha(pt)^2},$$

where $\alpha(t) \equiv 1$ in the double-exponential case, $\alpha(t) = t^{(1-\gamma)/(d+2-d\gamma)+o(1)}$ in the bounded case, and $1 \ll \alpha(t) = t^{o(1)}$ in the almost-bounded case.

In the almost-bounded case,

$$\chi = \min_{\psi \in \mathcal{C}(\mathbb{R}^d)} \Big[\frac{\varrho}{\mathrm{e}} \int_{\mathbb{R}^d} \mathrm{e}^{\psi(x)/\varrho}\,\mathrm{d}x - \lambda(\psi)\Big],$$

where $\lambda(\psi)$ is the top of the spectrum of $\Delta + \psi$ in $L^2(\mathbb{R}^d)$. This is the continuous analogue of the formula in the double-exponential case; it has the minimiser $\psi_\varrho(x) = \mathrm{const.} - \varrho^2|x|^2$ ($\Longrightarrow$ logarithmic Sobolev inequality).

Interpretation: The optimal shifted and rescaled potential (i.e., the one that contributes most mass to the moments of $U(t)$) is a perfect parabola, and the optimal shape of the properly rescaled and normalized solution $u(t,\cdot)$ is a perfect Gaussian density.

Theorems and heuristics in the almost-bounded case are analogous to the other cases, but proofs are considerably more difficult and technical.

Heavy-tailed potentials I

Consider the potential distribution (parameter $\alpha \in (d, \infty)$)

$$\text{Prob}(\xi(0) > r) = r^{-\alpha}, \qquad r \in [1, \infty), \qquad (\text{Pareto-distribution}).$$

Then the parabolic Anderson model possesses a.s. a solution $u(t, \cdot)$, but $U(t)$ has no moments. Nevertheless, $U(t)$ has an interesting behavior:

Weak Asmptotics for Pareto-Distributed Potentials [VAN DER HOFSTAD/MÖRTERS/SIDOROVA 2006]. If the potential ξ is Pareto-distributed with parameter $\alpha > d$, we have the weak convergence

$$\lim_{t\to\infty} \text{Prob}\Big(\Big(\frac{t}{\log t}\Big)^{-\frac{d}{\alpha-d}} \frac{1}{t} \log U(t) \leq x\Big) = \exp\big(-\mu x^{d-\alpha}\big),$$

where $\mu \in (0, \infty)$ is some suitable, explicit constant.

- Some a.s. limsup and liminf results for $\log(\frac{1}{t}\log U(t))$ are also contained in [HMS06].
- $\frac{1}{t}\log U(t)$ has the same weak asymptotics as the maximum of t^d independent Pareto $(\alpha - d)$-distributed random variables. Apparantly, the potential's random fluctuations dominate the smoothing effect of Δ^{d}.
- [HMS06] also contains analogous results for stretched-exponential upper tails (Weibull distribution).

Heavy-tailed potentials II

It is known that large values of a sum of i.i.d. heavy tailed random variables are most easily realised by having just one of the values extremely large (and the others of moderate size). This principle can be translated to the solution of the parabolic Anderson model:

Complete Localisation for Pareto-Distributed Potentials [K./MÖRTERS/SIDOROVA 2006]. If the potential ξ is Pareto-distributed with parameter $\alpha > d$, there is a process $(Z_t)_{t>0}$ in Z^d such that

$$\lim_{t\to\infty} \frac{u(t, Z_t)}{U(t)} = 1 \qquad \text{in probability.}$$

Furthermore, $Z_t(\log t/t)^{\alpha/(\alpha-d)}$ converges in distribution to some non-degenerate random variable.

- Hence, there is precisely one intermittent island for Pareto-distributed potentials in the parabolic Anderson model. Such a strong localisation could not be established for other potentials yet.
- The localisation statement is not true in almost sure sense, since the process $(Z_t)_{t>0}$ jumps.
- The proof uses results from [HMS06] (extreme value theory, asymptotics for $\frac{1}{t}\log U(t)$) and [GKM06] (spectral decomposition approach).