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# ON THE HOMOLOGY OF CONFIGURATION SPACES

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## 1. INTRODUCTION

1.1 By the $k$-th configuration space of a manifold $M$ we understand the space $C^k(M)$ of subsets of $M$ with cardinality $k$. If $\tilde{C}^k(M)$ denotes the space of $k$-tuples of distinct points in $M$, i.e. $\tilde{C}^k(M) = \{(z_1, \ldots, z_k) \in M^k \mid z_i \neq z_j \text{ for } i \neq j\}$, then $C^k(M)$ is the orbit space of $\tilde{C}^k(M)$ under the permutation action of the symmetric group $\Sigma_k$,

$$\tilde{C}^k(M) \to \tilde{C}^k(M)/\Sigma_k = C^k(M).$$

Configuration spaces appear in various contexts such as algebraic geometry, knot theory, differential topology or homotopy theory. Although intensively studied their homology is unknown except for special cases, see for example [1, 2, 7, 8, 9, 12, 13, 14, 18, 26] where different terminology and notation is used.

In this article we study the Betti numbers of $C^k(M)$ for homology with coefficients in a field $\mathbb{F}$. For $\mathbb{F} = \mathbb{F}_2$ the rank of $H_*(C^k(M); \mathbb{F}_2)$ is determined by the $\mathbb{F}_2$-Betti numbers of $M$, the dimension of $M$, and $k$. Similar results were obtained by Löffler–Milgram [17] for closed manifolds. For $\mathbb{F} = \mathbb{F}_p$ or a field of characteristic zero the corresponding result holds in the case of odd-dimensional manifolds; it is no longer true for even-dimensional manifolds, not even for surfaces, see [5], [6], or 5.5 here. Our methods do not suffice to determine either the product structure or the Steenrod operations.

1.2 The plan is to describe $H_* C^k(M)$ as part of the homology of a much larger space

$$C(M, M_0; X) = \left( \coprod_{k \geq 1} \tilde{C}^k(M) \underset{\Sigma_k}{\times} X^k \right) \Big/ \approx$$

where $M_0$ is a submanifold, $X$ a space with basepoint $x_0$, and $\approx$ is generated by $(z_1, \ldots, z_k; x_1, \ldots, x_k) \approx (z_1, \ldots, z_{k-1}; x_1, \ldots, x_{k-1})$ if $z_k \in M_0$ or $x_k = x_0$. Such spaces of labeled configurations occur in [4, 22] as models for mapping spaces. We will need the case $X = S^n$.

1.3 To formulate our results we introduce some notation. All manifolds $M$ are smooth, compact, and have a fixed dimension $m$. The submanifolds $M_0$ are compact and of arbitrary dimension, and possibly empty. $X$ is a CW complex with basepoint $x_0$. Let $\mathbb{F}$ be either the field $\mathbb{F}_p$ with $p$ elements, or a field of characteristic zero. $H_*()$ will always mean homology with coefficients in $\mathbb{F}$, and $\beta_q = \dim_{\mathbb{F}} H_q(M, M_0; \mathbb{F})$ is the $q$-th Betti number. $H_*$ can also stand for any graded $\mathbb{F}$-module such that $\beta_q = \dim_{\mathbb{F}} H_q$ is finite and $H_q = 0$ for $q > m$. Let $n \geq 1$ be an integer; unless $\mathbb{F}$ is $\mathbb{F}_2$, we require $m + n$ to be odd throughout this article.

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1.4 The first result describes the homology of $C(M, M_0; S^n)$ in terms of $H_*(M, M_0)$. For any graded $\mathbb{F}$-module $H_*$ define

$$\mathscr{C}(H_*; n) = \bigotimes_{q=0}^{m} \bigotimes^{\beta_q(H_*)} H_*\Omega^{m-q}S^{m+n}$$

Later we will give a more intrinsic definition in terms of Dyer–Lashof operations.

THEOREM A. *There is an isomorphism of graded vector spaces*

$$\theta: H_*C(M, M_0; S^n) \cong \mathscr{C}(H_*(M, M_0); n) \qquad \blacksquare$$

In the light of 2.5 below this should be compared with [15]. The isomorphism $\theta$ depends on the choice of a handle decomposition; it is natural for embeddings preserving the handle decompositions.

1.5 Each term $H_*\Omega^{m-q}S^{m+n}$ of $\mathscr{C} = \mathscr{C}(H_*; n)$ is an algebra with weights associated to its generators. This yields a product filtration $F_k\mathscr{C}(H_*; n)$ of $\mathscr{C}(H_*; n)$. On the other hand, the space $C(n) = C(M, M_0; S^n)$ has a filtration $F_kC$ given by the length of configurations. The filtration is known to split stably, see [4, 10, 25], and therefore $\tilde{H}_*C(n) \cong \bigoplus_k \tilde{H}_* (F_kC(n)/F_{k-1}C(n))$. The length filtration of $H_*C(n)$ and the weight filtration of $\mathscr{C}(H_*; n)$ agree via the isomorphism $\theta$.

THEOREM B. *There are isomorphisms of graded vector spaces*

$$\theta_k: H_*F_kC(M, M_0; S^n) \cong F_k\mathscr{C}(H_*(M, M_0); n) \qquad \blacksquare$$

1.6 Thus $\mathscr{D}_k(H_*; n) = F_k\mathscr{C}(H_*; n)/F_{k-1}\mathscr{C}(H_*; n)$ is the homology of the quotient space $F_kC(n)/F_{k-1}C(n)$, known as the $k$-adic construction $D_k(M, M_0; S^n)$. There is a vector bundle

$$\eta^k: \tilde{C}^k(M) \underset{\Sigma_k}{\times} \mathbb{R}^k \to C^k(M),$$

and $D_k(M; S^n)$ is the Thom space of its $n$-fold sum. By the Thom isomorphism we finally obtain the homology of $C^k(M)$ in terms of the homology of $M$, the dimension of $M$, and the number $k$. $\sigma^{-t}$ denotes the $t$-th desuspension as defined in 4.3 here.

THEOREM C. *For n even there are isomorphisms of graded vector spaces*

$$\bar{\theta}_k: H_*C^k(M) \cong \sigma^{-kn}\mathscr{D}_k(H_*(M); n) \qquad \blacksquare$$

Write $\mathbb{F}(-1)$ for the $\Sigma_k$-module $\mathbb{F}$ with $\Sigma_k$-action given by $\pi(1) = (-1)^{\text{sign}(\pi)}$ for $\pi$ in $\Sigma_k$. Then $H_*(C^k(M); \mathbb{F}(-1))$ means the homology of the chain complex $(S_*\tilde{C}^k(M)) \otimes_{\Sigma_k} \mathbb{F}(-1)$ where $S_*(.)$ denotes the singular chain complex. The proofs of the isomorphisms above apply to give an isomorphism between $H_*(C^k(M); \mathbb{F}(-1))$ and $\sigma^{-kn}\mathscr{D}_k(H_*(M); n)$ when $n$ is odd and $m$ is even.

1.7 The article is organized as follows. In Section 2 we recall the basic properties of $C(M, M_0; S^n)$. Section 3 contains a proof of Theorem A for $n \geq 2$. The proof of the remaining case $n = 1$ is given in Section 4, together with the proof of Theorem B. Section 5 contains some explicit examples.

## 2. THE CONFIGURATION SPACES

In this section $X$ can be an arbitrary CW complex with basepoint $x_0$. We list some properties of the bifunctor $C$.

2.1 *Invariance.* The homotopy type of $C(M, M_0; X)$ is an invariant of the homotopy type of $(X, x_0)$ and of the (relative) isotopy type of $(M, M_0)$. For example, if we extend the definition to open manifolds, the inclusion $C(M - \partial M; X) \to C(M; X)$ is a homotopy equivalence.

2.2 *Filtration.* $C = C(M, M_0; X)$ is filtered by closed subspaces

$$F_k C(M, M_0; X) = C_k(M, M_0; X) = \left( \coprod_{j=1}^{k} \tilde{C}^j(M) \underset{\Sigma_j}{\times} X^j \right) \Big/ \approx .$$

$F_0 C$ is the basepoint, and $F_1 C = (M/M_0) \wedge X$. The inclusions $F_{k-1} C \to F_k C$ are cofibrations, see [20; Th. 7.1]. Their cofibres are denoted by $D_k = D_k(M, M_0; X)$ and called the $k$-adic construction.

2.3 *Stable splittings and Hopf maps.* The filtration has a stable splitting, i.e. there is a natural equivalence for connected $X$

$$Q(C(M, M_0; X)) \to \prod_{k=1}^{\infty} Q(D_k(M, M_0; X)),$$

where $Q = \Omega^\infty S^\infty$. For $M = \mathbb{R}^m$ and $M_0 = \phi$ this is the Snaith splitting of [25], compare 2.5. The general case is proved in [4, 10]. Composing with the stabilization map $C \to Q(C)$ on the left side, and with projection onto the first factor $Q(D_1) = Q(M/M_0 \wedge X)$ on the right side yields the first Hopf map

$$h = h(M, M_0) \colon C(M, M_0; X) \to Q(M/M_0 \wedge X).$$

2.4 *Quasifibrations.* If $N \subset M$ is a submanifold of codimension zero, and if $N/(N \cap M_0)$ or $X$ is connected, then

$$C(N, N \cap M_0; X) \to C(M, M_0; X) \xrightarrow{q} C(M, N \cup M_0; X)$$

is a quasifibration. For $X = S^0$ this is proved in [22; Prop. 3.1]. For general $X$ one proceeds in the same way; only for the sake of completeness we sketch the various steps.

(1) Filter the base $B = C(M, N \cup M_0; X)$ by $B_k = C_k(M, M \cup M_0; X)$, and filter the total space $E = C(M, M_0; X)$ by $E_k = q^{-1}(B_k)$; denote the fibre by $F = C(N, N \cap M_0; X)$.

(2) Observe, that for each $k$ there is a homeomorphism

$$t_k \colon E_k - E_{k-1} \to (B_k - B_{k-1}) \times F \text{ over } (B_k - B_{k-1}).$$

(3) Let $U$ be a closed tubular neighborhood of the pair $(N, N \cap M_0)$, and $r \colon M \to M$ an isotopy leaving $M_0$, $N$ and $U$ invariant and retracting exactly $U$ into $N \cup M_0$; then define

$$U_k = \{ b = [z_1, \ldots, z_k; x_1, \ldots, x_k] \in B_k \mid \text{at least one } z_i \text{ lies in } U \}$$

for each $k$. This is a neighborhood of $B_{k-1}$ in $B_k$; $r$ induces a retraction $\bar{r}_k \colon q^{-1}(U_k) \to q^{-1}(B_{k-1}) = E_{k-1}$ and a retraction $r_k \colon U_k \to B_{k-1}$, and $q \circ \bar{r}_k = r_k \circ q$.

(4) Over every $b \in U_k$ separately the restriction to fibres

$$\varphi_b: F \xrightarrow[\cong]{t_k} q^{-1}(b) \xrightarrow{\bar{r}_k} q^{-1}(r(b)) \xrightarrow[\cong]{t_{k-1}^{-1}} F$$

is homotopic to the identity. To see this we write

$$b = [z_1, \ldots, z_1', \ldots, ; x_1, \ldots, x_1', \ldots,] \text{ such that } z_1, z_2, \ldots, \in M - U$$

and $z_1', z_2', \ldots \in U - N \cup M_0$. Let $f = [z_1'', \ldots, ; x_1'', \ldots,] \in F$. We have

$$t_k(f) = [z_1, \ldots, z_1', \ldots, z_1'', \ldots, ; x_1, \ldots, x_1', \ldots, x_1'', \ldots],$$

and

$$\bar{r}_k t_k(f) = [r(z_1), \ldots, r(z_1'), \ldots, r(z_1''), \ldots; x_1, \ldots, x_1', \ldots, x_1'', \ldots],$$

and

$$t_{k-1}^{-1} \bar{r}_k t_k(f) = [r(z_1'), \ldots, r(z_1''), \ldots; x_1', \ldots, x_1'', \ldots] = \varphi_b(f).$$

Note that the $z_i'$ and $x_i'$ depend on $b$ only; hence moving $r(z_i')$ in $N$ to $N \cap M_0$, or moving their labels $x_i'$ in $X$ to $x_0$ defines a homotopy of $\varphi_b$, which ends with $\bar{r}_k | F$. Since $\bar{r}_k$ is homotopic to the identity, so is $\varphi_b$. It follows from [16; 2.10, 2.15, 5.2] that $q$ is a quasifibration.

2.5 *Section spaces.* Assume $W$ is a $m$-manifold without boundary and containing $M$. For example, $W = M$ if $M$ is closed, or $W = W \cup \partial M \times [0, 1[$ if $M$ has boundary. Let $\dot{T}W$ be the fibrewise compactification of the tangent bundle of $W$. $\Gamma(W - M_0, W - M; X)$ denotes the space of cross sections of $\dot{T}W_{\hat{w}}(W \times X)$ which are defined an $W - M_0$ and are inifinity on $W - M$. There is a (weak) equivalence $C(M, M_0; X) \to \Gamma(W - M_0, W - M; X)$ if $M/M_0$ or $X$ is connected, see [22: Th. 1.4] or [4; Prop. 2]. For a handle of index $q$, i.e. for $M = [0, 1]^m$ and $M_0 = [0, 1]^{m-q} \times \partial([0, 1]^q)$, this means $C(M, M_0; X) \simeq \Omega^{m-q} S^m X$ if $X$ is connected, which is a special case of Theorem A. The case $q = 0$ is the approximation theorem of [3, 20, 24].

2.6 *Vector bundles over configuration spaces.* For every $k$ there is a vector bundle

$$\eta^k = \eta^k(M): \tilde{C}^k(M) \underset{\Sigma_k}{\times} \mathbb{R}^k \to C^k(M).$$

It has finite order $n_k$; lower and upper bounds follow from the results in [11], where $n_k$ was computed for $M = \mathbb{R}^m$. Let $\eta_0^k$ be the restriction of $\eta^k$ to $C^k(M \mid M_0) = \tilde{C}^k(M \mid M_0)/\Sigma_k$ where $\tilde{C}^k(M \mid M_0) = \{(z_1, \ldots, z_k) \in \tilde{C}^k(M) \mid z_i \in M_0 \text{ for at least one } i\}$. The relative Thom space of $n$ times the pair $(\eta^k, \eta_0^k)$ is homeomorphic to $D_k(M, M_0; S^n)$, see [23; Th. 1.3.2]. Thus, if $M_0 = \phi$, we have $D_k(M; S^{n_k}) \cong S^{k \cdot n_k} C^k(M)_+$. More generally, there is a periodicity $D_k(M; S^{n+n_k}) \cong S^{kn_k} D_k(M; S^n)$ for any $n \geq 1$.

### 3. AN EMBEDDING IN HOMOLOGY

3.1 The purpose of this section is to prove the Lemma. Assume $n \geq 2$.

(a) *There is an isomorphism of graded vector spaces*

$$\theta: H_* C(M, M_0; S^n) \cong \bigotimes_{q=0}^{m} \bigotimes_{j=1}^{\beta_q} H_* \Omega^{m-q} S^{m+n}$$

(b) *The first Hopf map induces a monomorphism*

$$h_*: H_*C(M, M_0; S^n) \to H_*Q(M/M_0 \wedge S^n). \qquad \blacksquare$$

This lemma implies Theorem A for $n \geq 2$. We remark that it is also correct for $n = 1$, but for technical reasons we postpone this case until 4.5. As the proof will show, $\theta$ depends on the choice of a handle decomposition. It is natural for embeddings which respect the handle decompositions. $\theta$ will preserve the grading which is important in Lemma 4.2.

3.2 *Proof of Lemma* 3.1 First we will prove the absolute case $M_0 = \phi$ by induction on a handle decomposition of $M$. If $M$ is a disjoint union $M_1 \amalg M_2$, then $C(M; X) \cong C(M_1; X) \times C(M_2; X)$. Thus we can restrict to connected manifolds and start with $M$ an $m$-disc $D^m$. By 2.5. $C(D^m; S^n) \simeq \Omega^m S^{m+n}$, and (a) is obvious. The assertion (b) is proved in [12; p. 226]; here we need the hypotheses that $m + n$ is odd if $\mathbb{F} \neq \mathbb{F}_2$.

Assume (a) and (b) hold for $M$. If $\bar{M} = M \cup D$ with $D \cong [0, 1]^m$ a handle of index $q$ in $M$, i.e. $D \cap M \cong [0, 1]^{m-q} \times \partial[0, 1]^q$, we can assume $q \geq 1$, because $M$ is connected. There is a cofibration

$$M \to \bar{M} \to (\bar{M}, M) \simeq (S^q, *)$$

and the alternative:

I. $H_q(\bar{M}) \to \tilde{H}_q(S^q)$ is epic, i.e. $\beta_q(\bar{M}) = \beta_q(M) + 1$;
II. $H_q(\bar{M}) \to \tilde{H}_q(S^q)$ is zero, i.e. $\beta_{q-1}(\bar{M}) = \beta_{q-1}(M) - 1$.

3.3 *Case I.* The diagram

$$\begin{array}{ccc} C(M; S^n) & \xrightarrow{h(M)} & Q(M_+ \wedge S^n) \\ \downarrow & & \downarrow \\ C(\bar{M}; S^n) & \xrightarrow{h(\bar{M})} & Q(\bar{M}_+ \wedge S^n) \\ \downarrow & & \downarrow \\ \Omega^{m-q} S^{m+n} \simeq C(\bar{M}, M; S^n) & \xrightarrow{h(\bar{M}, M)} & Q(\bar{M}/M \wedge S^n) \simeq Q(S^{q+n}) \end{array}$$

is commutative by 2.3, the left column is a quasifibration by 2.4, the right column is a fibration. Since $H_*\bar{M} \to H_*S^q$ is epic in this case, so is $H_*Q(\bar{M}_+ \wedge S^n) \to H_*Q(S^{q+n})$. Thus the Serre spectral sequence on the right side collapses. $h(M)_*$ is monic by the inductive assumption, which forces the Serre spectral sequence on the left side to collapse also. Hence $H_*C(\bar{M}; S^n) \cong H_*C(M; S^n) \otimes H_*\Omega^{m-q}S^{m+n}$ which proves assertion (a) for $\bar{M}$. On the $E^2$-level of the spectral sequence $h(\bar{M})_*$ corresponds to $h(M)_* \otimes h(\bar{M}, M)_*$. Since both $h(M)_*$ and $h(\bar{M}, M)_*$ are monic, this gives (*b*) for $\bar{M}$.

3.4 *Case II.* In the diagram

$$\begin{array}{ccc} \Omega^{m-q+1} S^{m+n} \simeq \Omega C(\bar{M}, M; S^n) & \xrightarrow{\Omega h(\bar{M}, M)} & \Omega Q(\bar{M}/M \wedge S^n) \simeq Q(S^{q-1+n}) \\ \downarrow & & \downarrow \\ C(M; S^n) & \xrightarrow{h(M)} & Q(M_+ \wedge S^n) \\ \downarrow & & \downarrow \\ C(\bar{M}; S^n) & \xrightarrow{h(\bar{M})} & Q(\bar{M}_+ \wedge S^n) \end{array}$$

$H_*Q(S^{q-1+n}) \to H_*Q(M_+ \wedge S^n)$ is monic, because $H_*S^{q-1} \to H_*M$ is in this case. Note that this is the only case where we need $n \geq 2$ to ensure the triviality of the local coefficient system. Thus the Serre spectral sequence on the right side collapses. $\Omega h(\bar{M}, M)$ may be replaced by $\Omega^{m-q+1}E$ where $E$ is the stabilization map $S^{m+n} \to Q(S^{m+n})$. Therefore $\Omega h(\bar{M}, M)_*$ is monic, the Serre spectral sequence on the left side collapses and $H_*C(M; S^n) \cong H_*C(\bar{M}; S^n) \otimes H_*\Omega^{m-q+1}S^{m+n}$. Comparing the Euler–Poincaré series proves (a) for $\bar{M}$. To see that $h(\bar{M})_*$ is monic, observe that (1) both fibrations are principal, (2) $H_*C(\bar{M}; S^n) \cong H_*C(M; S^n) \otimes_R \mathbb{F}$, $R = H_*\Omega^{m-q+1}S^{m+n}$, and (3) $H_*Q(\bar{M} \wedge S^n) \cong H_*Q(M_+ \dot{\wedge} S^n) \otimes_{R'} \mathbb{F}$, $R' = H_*Q(S^{q-1+n})$. Then the result follows from naturality.

3.5 To treat the relative case we can assume that $M_0$ is part of an open collar, $M_0 = (\partial M \cap M_0) \times [0, 1[$. To see this, we first replace $M_0$ by a closed tubular neighborhood $M_0'$. Next we remove its interior and obtain $M_0'' = M_0' - \operatorname{int} M_0'$, which lies in the boundary of $M'' = M - \operatorname{int} M_0'$. Then we attach an open collar to form $M''' = M'' \cup (\partial M'' \times [0, 1[)$, and set $M_0''' = M_0'' \times [0, 1[$. During this procedure the homotopy type of $C$ was not changed by 2.1 and 2.5, and the last pair $(M''', M_0''')$ has the desired form.

Lemma 3.1 (a) and (b) will be proved by induction on a handle decomposition of $M_0 \cap \partial M$. The start of the induction was the absolute case $M_0 = \phi$.

Assume that (a) and (b) hold for $(M, M_0)$. Let $\bar{M}_0$ be $M_0 \cup D$ with $D \cong [0, 1]^m$ of the form $(D \cap \partial M) \times [0, 1]$ such that $D \cap \partial M \cong [0, 1]^{m-1} \times$ is a handle of dimension $m - 1$ and index $q (0 \leq q \leq m - 1)$ in $\bar{M}_0 \cap \partial M$ i.e. $D \cap \partial M \cap M_0 \cong [0, 1]^{m-1-q} \times \partial [0, 1]^q \times 0$. There is a cofibration

$$S^q \simeq (D, D \cap M_0) \to (M, M_0) \to (M, \bar{M}_0)$$

and again an alternative:

III. $H_q(S^q) \to H_q(M, M_0)$ is monic, i.e. $\beta_q(M, \bar{M}_0) = \beta_q(M, M_0) - 1$;

IV. $H_q(S^q) \to H_q(M, M_0)$ is zero, i.e. $\beta_{q+1}(M, \bar{M}_0) = \beta_{q+1}(M, M_0) + 1$.

3.6 *Case III.* In the diagram

$$\begin{array}{ccc}
\Omega^{m-1}S^{m+n} \simeq C(D, D \cap M_0; S^n) & \xrightarrow{h(D,\, D \cap M_0)} & Q(D/(D \cap M_0) \wedge S^n) \simeq Q(S^{q+n}) \\
\downarrow & & \downarrow \\
C(M, M_0; S^n) & \xrightarrow{H(M,\, M_0)} & Q(M/M_0 \wedge S^n) \\
\downarrow & & \downarrow \\
C(M, \bar{M}_0; S^n) & \xrightarrow{h(M,\, \bar{M}_0)} & Q(M/\bar{M}_0 \wedge S^n)
\end{array}$$

$H_*Q(S^{q+m}) \to H_*Q(M/M_0 \wedge S^n)$ is monic, since $H_*S^q \to H_*(M, M_0)$ is monic in this case. Also $h(D, D \cap M_0)$ is monic. The arguments to show (a) and (b) for $(M, \bar{M}_0)$ are now similar to case II.

3.7 *Case IV.* Remove $D' = ]0, 1[^{m-q-1} \times ]0, 1[^q \times [0, 1[ \subset D$ from $M$, $N = M - D'$, and consider the diagram

$$\begin{array}{ccc} C(N, N\cap \bar{M}_0; S^n) & \xrightarrow{h(N, N\cap \bar{M}_0)} & Q(N/(N\cap \bar{M}_0)\wedge S^n) \\ \downarrow & & \downarrow \\ C(M, \bar{M}_0; S^n) & \xrightarrow{h(M, \bar{M}_0)} & Q(M/\bar{M}_0 \wedge S^n) \\ \downarrow & & \downarrow \\ C(M, \bar{M}_0\cup N; S^n) & \xrightarrow{h(M, \bar{M}_0\cup N)} & Q(M/\bar{M}_0 \cup N)\wedge S^n). \end{array}$$

The pair $(N, N\cap \bar{M}_0)$ is isotopic to $(M, M_0)$, hence $h(N, N\cap \bar{M}_0)_*$ is monic by inductive assumption. $H_*Q(N/(N\cap \bar{M}_0)\wedge S^n)\to H_*Q(M/\bar{M}_0)\wedge S^n)$ is monic in this case. Therefore the Serre spectral sequence on both sides collapses. Since $(M, \bar{M}_0\cup N)$ is isotopic to the handle $([0,1]^m, [0,1]^{m-q-1}\times \partial([0,1]^q\times[0,1]))$ of index $q+1$, $C(M, \bar{M}_0\cup N; S^n)\simeq \Omega^{m-q-1}S^{m+n}$. The assertions (a) and (b) follow as in case I.

## 4. AN ALGEBRAIC FILTRATION OF $H_*C(M, M_0; S^n)$

Throughout this section the pair $(M, M_0)$ will be kept fixed, and we write $C(n)$ for $C(M, M_0; S^n)$, and $H_*$ for $H_*(M, M_0)$. The proofs will be given for the field $\mathbb{F} = \mathbb{F}_2$ as the other cases are quite similar.

4.1 The graded algebra $\mathscr{C}(H_*; n)$ was defined in 1.4 as

$$\mathscr{C}(H_*; n) = \bigotimes_{q=0}^{m} \bigotimes^{\beta_q} H_*(\Omega^{m-q}S^{m+n}),$$

and thus depends only on the numbers $m$, $n$ and $\beta_0, \ldots, \beta_m$. The intrinsic definition is as follows. First, we introduce for each $\alpha\in H_q$ a generator $u_\alpha$, and set as degree and weight

(1) $|u_\alpha| = |\alpha| + n$,
(2) $\omega(u_\alpha) = 1$.
Secondly, for each $u_\alpha$ and index $I = (i_1, i_2, \ldots i_r)$ there is an additional generator $Q_I u_\alpha$ if the condition
(3) $0 < i_1 \leq i_2 \leq \ldots \leq i_r < m - |\alpha|$
holds. $Q_I$ stands for $Q_{i_1}Q_{i_2}\cdots Q_{i_r}$, and the $Q_i$ are the Dyer–Lashof operations. We have
(4) $|Q_I u_\alpha| = i_1 + 2i_2 + 4i_3 + \ldots + 2^{r-1}_{i_r} + 2^r(|\alpha| + n)$,
(5) $\omega(Q_I u_\alpha) = 2^{\|I\|} = 2^r$.

Note that there are no additional generators $Q_I u_\alpha$ if $\alpha\in H_m$ or $\alpha\in H_{m-1}$.

These generators are subject to the following relations
(6) $u_{\alpha+\beta} = u_\alpha + u_\beta$,
(7) $Q_I u_{\alpha+\beta} = Q_I u_\alpha + Q_I u_\beta$,
(8) $u_\alpha^2 = 0$ if $|\alpha| = m$.

Then $\mathscr{C}(H_*; n)$ is the associative and commutative $\mathbb{F}$-algebra generated by all $u_\alpha$ and $Q_I u_\alpha$, modulo the relations (6)–(8). The degree and weight are extended by $|v_1\cdot v_2| = |v_1| + |v_2|$, and $\omega(v_1\cdot v_2) = \omega(v_1) + \omega(v_2)$. Using the weight function we filter $\mathscr{C}(H_*; n)$ by defining $F_k\mathscr{C}(H_*; n) = \mathscr{C}_k(H_*; n)$ to be the submodule spanned by the monomials of weight at most $k$.

If $H_* = \mathbb{F}_2$ is concentrated in degree $* = q$, then $\mathscr{C}(H_*; n)\cong H_*\Omega^{m-q}S^{m+n}$, see [12]. For example, if $(M, M_0)$ is a handle of index $q$, then $C(M, M_0; S^n)\simeq \Omega^{m-q}S^{m+n}$. Note that in this case the generators $Q_I u_\alpha$ and their weight were first defined using configuration spaces and their length filtration.

4.2 A handle decomposition of $(M, M_0)$ provides a vector space basis of $H_* = H_*(M, M_0)$, and thus following 4.1 an algebra basis of $\mathscr{C}(H_*; n)$. Moreover, Lemma 3.1 yields an isomorphism of graded vector spaces,

$$\theta: \mathscr{C}(H_*; n) \to H_* C(n).$$

In order to prove Theorem B we will show that in fact any graded isomorphism must preserve the respective filtrations $\mathscr{C}_k(H_*; n)$ and $H_* C_k(n)$ through a range increasing with $n$. The stable splitting of 2.3 and the periodicity of 2.6 will then complete the proof.

LEMMA. *Let $r$ be an integer such that $n > rm \geq 2$. Then the following hold for all $k \leq r$:*

(a) *$\theta$ maps $\mathscr{C}_k(H_*; n)$ to $H_* C_k(n)$;*
(b) *both $\mathscr{C}_k(H_*; n)$ and $H_* C_k(n)$ vanish in degrees $\geq n(k+1)$;*
(c) *the restriction $\theta_k: \mathscr{C}_k(H_*; n) \to H_* C_k(n)$ is an isomorphism.*

*Proof.* The dimension of $Q_i^k(v) = Q_i \ldots Q_i(v)$ ($k$ times) is $|Q_i^k(v)| = (2^k - 1)i + 2^k|v|$. The elements in $\mathscr{C}_{2^k}(H_*; n)$ of maximal degree are spanned by elements of the form $Q_{m-i-1}^k(u_{n+i})$ for $i$ maximal with $1 \leq m-1$ and $\beta_i \neq 0$, as one can see by inspection. In general, if $k = \sum_{\kappa \in K} 2^\kappa$ is the 2-adic expansion of $k$, then the elements of maximal degree in $\mathscr{C}_k(H_*; n)$ are spanned by elements of the form $v = \prod_{\kappa \in K} Q_{m-i-1}^\kappa(u_{n+i})$ for $i$ maximal again. Thus

$$\begin{aligned} |v| &= \sum_{\kappa \in K} ((2^\kappa - 1)(m-i-1) + 2^\kappa(n+i)) \\ &= \sum_{\kappa \in K} (2^\kappa(m+n-1) + i + 1 - m) \\ &= k(m+n-1) + \operatorname{card}(K)(i+1-m) < k(n+i) \end{aligned}$$

gives a bound for this maximal degree. Since $i < m$, $k \leq r$ and $km < n$, this gives $|v| < n(k+1)$. But $H_* C(n) \cong \bigoplus_{j \geq 1} \tilde{H}_* D_j(n)$ by the stable splitting 2.3, and $D_j(n)$ is $(jn-1)$-connected. Therefore $\theta(v)$ must lie in $H_* C_j(n)$ for some $j < k+1$, which proves (a).

To prove (b) recall from 2.6 that $D_j(n)$ is the Thom space of a $nj$-dimensional vector bundle over the $mj$-dimensional base $C^j(M)$. Hence $H_* D_j(n) = 0$ for $* > (m+n)j$. This implies $H_* C_k(n) \cong \bigoplus_{j \leq k} H_* D_j(n) = 0$ for $* > (m+n)k$. Furthermore, $\mathscr{C}_k(n)$ vanishes in dimensions $* > (m+n)k$ by construction. As $n(k+1) > (m+n)k$, both assertions of (b) follow.

Consider the diagram obtained by restriction

$$\begin{array}{ccc} \mathscr{C}(H_*; n) & \xrightarrow[\cong]{\theta} & H_* C(n) \\ \cup & & \cup \\ \mathscr{C}_k(H_*; n) & \xrightarrow[\theta_k]{} & H_* C^\kappa(n). \end{array}$$

For part (c) we have to show the surjectivity of $\theta_k$, i.e. $v = \theta^{-1}(w)$ is in $\mathscr{C}_k(H_*; n)$ for any $w \in H_*C(n)$ with $|w| = * < n(k+1)$. But this is the case, for $v$ represents zero in $\mathscr{C}(H_*; n)/\mathscr{C}_k(H_*; n)$ which vanishes in degrees $* < n(k+1)$ by construction. ■

4.3 Let $\sigma^k A$ denote the graded module $A$ with the degrees of the new elements $\sigma^k(a)$ being raised by $k$. Denote by $\mathscr{D}_k(H_*; n)$ the graded vector space $\mathscr{C}_k(H_*; n)/\mathscr{C}_{k-1}(H_*; n)$.

LEMMA. *If* $\mathbb{F} = \mathbb{F}_2$ *there is an isomorphism of vector spaces*

$$t\colon \sigma^k \mathscr{D}_k(H_*; n) \to \mathscr{D}_k(H_*; n+1).$$

*For general* $\mathbb{F}$ *there is an isomorphism* $\sigma^{2k}\mathscr{D}_k(H_*; n) \cong \mathscr{D}_k(H_*; n+2)$.

*Proof.* A basis of $\mathscr{D}_k(H_*; n)$ is given by the monomials $v = Q_{I_1}(u_{i_1})\cdot \ldots \cdot (Q_{I_r})(u_{i_r})$ with $k = 2^{\|I_1\|} + \ldots + 2^{\|I_r\|}$. Note that the degree of $\sigma^{2^{\|I\|}}(Q_I(u_i))$ and $Q_I(\sigma(u_i))$ agree, where $\sigma(u_i) = u_{i+1}$. Therefore $t(\sigma^k(v)) = Q_{I_1}(u_{i_1+1})\cdot \ldots \cdot Q_{I_r}(U_{i_r}+1)$ defines an isomorphism. The case $\mathbb{F} \neq \mathbb{F}_2$ is analogous. ■

4.4 *Proof of Theorem B.* If $n$ and $k$ are such that $n > km \geq 2$, then the isomorphism $\theta_k$: $\mathscr{C}_k(H_*; n) \to H_*C_k(n)$ of Lemma 4.2 induces an isomorphism $\bar{\theta}_k\colon \mathscr{D}_k(H_*; n) \to \tilde{H}_*D_k(n)$. For arbitrary $n, k \geq 1$ choose $n'$ such that

(1) $n + n' > km$, and

(2) $n + n'$ is a multiple of the order of the bundle $\eta^k$. We exclude the trivial case $m = k = 1$ and assume in addition that $km \geq 2$. By (2) and 2.6 there is an isomorphism

$$d\colon \sigma^{kn'}\tilde{H}_*D_k(n) \to H_*D_k(n+n'),$$

Then an isomorphism $\bar{\theta}_k$ is defined by the following diagram,

$$\begin{array}{ccc}
\mathscr{D}_k(H_*; n) & \xrightarrow{\bar{\theta}_k} & \tilde{H}_*D_k(n) \\
\sigma^{-kn'}\big\downarrow \cong & & \cong \big\downarrow \sigma^{-kn'}(d) \\
\sigma^{-kn'}\mathscr{D}_k(H_*; n+n') & \xrightarrow[\sigma^{-kn'}(\bar{\theta}_k)]{\cong} & \sigma^{-kn'}H_*D_k(n+n')
\end{array}$$

where the lower $\bar{\theta}_k$ exists by (1). This implies Theorem B. ■

4.5 *Proof of Theorem A for* $n = 1$. The isomorphism $\theta$ is defined via the stable splitting 2.3, using the isomorphisms $\bar{\theta}_k$ of Theorem B. Here $\tilde{\mathscr{C}}$ means the elements of $\mathscr{C}$ in positive degrees.

$$\begin{array}{ccc}
\tilde{\mathscr{C}}(H_*; 1) & \xrightarrow{\theta} & \tilde{H}_*C(1) \\
\cong \big\downarrow & & \big\downarrow \cong \\
\bigoplus_{k\geq 1} \tilde{\mathscr{D}}_k(H_*; 1) & \xrightarrow[\bigoplus_{k\geq 1}\bar{\theta}_k]{\cong} & \bigoplus_{k\geq 1} \tilde{H}_*D_k(1)
\end{array}$$

■

Together with Lemma 3.1 this case completes the proof of Theorem A.

## 5. EXAMPLES

5.1 Let $M_g$ denote a closed, orientable surface of genus $g$. Take $\mathbb{F} = \mathbb{F}_2$ and $n = 2$. Since $\beta_q(M_g) = 1, 2g, 1$ for $q = 0, 1, 2$ and zero otherwise, we have by Theorem A

$$\begin{aligned} H_*C(M_g; S^n) &\cong H_*(S^{n+2}) \otimes H_*(\Omega S^{n+2})^{\otimes 2g} \otimes H_*(\Omega^2 S^{n+2}) \\ &\simeq \mathbb{F}_2[u]/u^2 \otimes \mathbb{F}_2[x_i | i = 1, \dots 2g] \otimes \mathbb{F}_2[y_j | j \geq 0] \end{aligned}$$

with

$$\begin{aligned} |u| &= n + 2, & \omega(u) &= 1, \\ |x_i| &= n + 1, & \omega(x_i) &= 1, \\ |y_i| &= (2^j - 1) + 2^j n, & \omega(y_j) &= 2^j. \end{aligned}$$

By Theorem B, a basis of $\tilde{H}_* D_k(M_g; S^n)$ consists of all monomials

$$f = u^e \cdot x_1^{a_1} \cdot \dots \cdot x_{2g}^{a_{2g}} \cdot y_1^{b_1} \cdot \dots \cdot y_r^{b_r}$$

for some $r \geq 0$, $e = 0, 1$ and $a_i, b_j \geq 0$, such that

$$\omega(f) = e + a_1 + \dots + a_{2g} + b_0 + 2b_1 + \dots + 2^r b_r = k.$$

For example, $\tilde{H}_q D_2(M_g; S^n)$ is determined by the following table.

| | basis | rank |
|---|---|---|
| $q = 2n$ | $y_0^2$ | 1 |
| $2n+1$ | $x_1 y_0, \dots, x_{2g} y_0, y_1$ | $2g + 1$ |
| $2n+2$ | $u y_0, x_1^2, \dots, x_i x_j, \dots, x_{2g}^2$ | $2g^2 + g + 1$ |
| $2n+3$ | $u x_1, \dots, u x_{2g}$ | $2g$ |

By Theorem C, the rank of $H_q C^2(M_g)$ is $1, 2g + 1, 2g^2 + g + 1$ and $2g$ for $q = 0, 1, 2, 3$. The next table gives $\tilde{H}_q D_3(M_g; S^n)$ and $H_{q-3n} C^3(M_g)$.

| | basis | rank |
|---|---|---|
| $q = 3n$ | $y_0^3$ | 1 |
| $3n+1$ | $y_0 y_1, x_i y_0^2 \quad (1 \leq i \leq 2g)$ | $2g + 1$ |
| $3n+2$ | $u y_0^2, x_i y_1, x_i x_j y_0 \quad (1 \leq i \leq j \leq 2g)$ | $2g^2 + 3g + 1$ |
| $3n+3$ | $u y_1, u x_i y_0, x_i x_j x_k \quad (1 \leq i \leq j \leq k \leq 2g)$ | $\frac{4}{3}g^3 + 2g^2 + \frac{8}{3}g + 1$ |
| $3n+4$ | $u x_i x_j \quad (1 \leq i \leq j \leq 2g)$ | $2g^2 + g$ |

The last table shows the rank of $H_q C^k(S^2)$ for $k \leq 10$.

| | $k = 1$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $q = 0$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | | | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 3 |
| 4 | | | | | 1 | 2 | 2 | 3 | 3 | 3 |
| 5 | | | | | 1 | 1 | 2 | 3 | 3 | 4 |
| 6 | | | | | | | 1 | 2 | 3 | 4 |
| 7 | | | | | | | | 1 | 2 | 3 |
| 8 | | | | | | | | | 1 | 2 |
| 9 | | | | | | | | | 1 | 1 |

5.2 Let $M$ be the complement of a knot in $\mathbb{R}^3$. Take $\mathbb{F} = \mathbb{F}_2$ and $n = 2$. Now $\beta_q(M) = 1, 1, 1$ for $q = 0, 1, 2$ and we have

$$H_*C(M; S^n) \cong H_*(\Omega S^{n+3}) \otimes H_*(\Omega^2 S^{n+3}) \otimes H_*(\Omega^3 S^{n+3})$$
$$\cong \mathbb{F}_2[x] \otimes \mathbb{F}_2[y_i | i \geq 0] \otimes \mathbb{F}_2[z_{ij} | i, j \geq 0]$$

with

$$|x| = n + 2 \qquad \omega(x) = 1$$
$$|y_i| = (2^i - 1) + 2^i(n + 1), \quad \omega(y_i) = 2^i$$
$$|z_{ij}| = 2^{i+j}(n + 2) - 2^i - 1, \quad \omega(z_{ij}) = 2^{i+j}$$

Proceeding as above gives the following table for $H_qC^2(M)$, and ranks of $H_qC^k(M)$ for $k = 2, 3, 4$.

| | rank $k = 2$ | 3 | 4 |
|---|---|---|---|
| $q = 0$ | 1 | 1 | 1 |
| 1 | 2 | 2 | 2 |
| 2 | 3 | 4 | 5 |
| 3 | 2 | 5 | 8 |
| 4 | 1 | 4 | 9 |
| 5 | | 2 | 8 |
| 6 | | 1 | 6 |
| 7 | | | 3 |
| 8 | | | 1 |

For example, a basis for $H_*C^2(M)$ is: $z_{00}^2$ in dimension $q = 0$; $y_0z_{00}$, $z_{10}$ in dimension $q = 1$; $xz_{00}$, $y_0^2$, $z_{01}$ in dimension $q = 2$; $xy_0$, $y_1$ dimension $q = 3$; and $x^2$ in dimension $q = 4$.

5.3 In this example we show that the results in Theorem A–C can be deduced by other means in case $M$ is a compact connected Lie group $G$. In this case $C(G; S^n)$ is homotopy equivalent to $\mathrm{map}(G; S^{n+g})$, the space of all maps from $G$ to $S^{n+g}$, $g = \dim G$, by 2.3. Let $\mathrm{map}_0(;)$ denote the space of based maps, and consider the diagram of evaluation fibrations

$$\begin{array}{ccc} \mathrm{map}_0(G; S^{n+g}) & \longrightarrow & \mathrm{map}_0(G; QS^{n+g}) \\ \downarrow & & \downarrow \\ \mathrm{map}(G; S^{n+g}) & \longrightarrow & \mathrm{map}(G; QS^{n+g}) \\ \downarrow & & \downarrow \\ S^{n+g} & \xrightarrow[E]{} & QS^{n+g} \end{array}$$

where $E$ is the stabilization. Evidently, the right-hand fibration is principal and has a section; thus the total space splits as $QS^{n+g} \times \mathrm{map}_0(G; QS^{n+g})$. The homology of $\mathrm{map}_0(G; S^{n+g})$ and $\mathrm{map}_0(G, QS^{n+g})$ is given in [15] as long as $n$ is sufficiently large, and $n + g$ is odd if $\mathbb{F} \neq \mathbb{F}_2$. In particular, the upper map in the diagram above is monic in homology. Thus, under these hypothesis, one has

$$H_*C(G; S^n) \cong \bigotimes_{q=0}^{g} \bigotimes_{j=1}^{\beta_g(G)} H_*(\Omega^{g-q} S^{n+g})$$

as given in Theorem A. Theorems B and C follow analogously.

5.4 The rational homology of $C(M \times \mathbb{R}; X)$ was determined in [13]. When $X$ is $S^n$ and $n+m+1$ is odd, the homology is given by the vector space isomorphism

$$H_* C(M \times \mathbb{R}; S^n) \cong \mathrm{Sym}[H_*(M) \otimes \tilde{H}_*(S^n)]$$

where Sym [ ] denotes the symmetric algebra. This last result is a special case of Theorem A when $n+m+1$ is even. Generally, one has the vector space isomorphism

$$H_* C(M \times \mathbb{R}; X) \cong \mathrm{Sym}[H_*(M) \otimes \sigma^{-m} L[\sigma^m \tilde{H}_*(X)]]$$

for path-connected $X$, where $\sigma$ is the suspension; $L[\,]$ is the free Lie algebra. There is a spectral sequence $E^2 = \mathrm{Tor}^R(\mathbb{Q}, \mathbb{Q})$ and $R = \mathrm{Sym}[H_*(M) \otimes \sigma^{-m} L[\sigma^m \tilde{H}_*(X)]]$ converging to $HC(M; SX)$. Note that $C(M \times \mathbb{R}; X) \simeq \Omega C(M; SX)$ by 2.5. There are non-trivial differentials when $X = S^n$ and $n+m$ is even.

5.5 In this section we recall some information from [5 or 6] concerning the rational homology of configurations in a punctured Riemann surface of genus $g$, $\bar{M}_g$. There we determine the rational homology of $C(\bar{M}_g; S^{2n})$. The relevant Serre spectral sequence (in cohomology) has an $E_2$-term given by $H^*(\Omega S^{2n+2})^{2g} \otimes H^*\Omega^2 S^{2n+2}$. There is a non-trivial (integral) differential which hits twice the form $\sum_1^g x_{2i-1} x_{2i}$ where the $x_i$ run over a choice of generators for $H^{2n+1}(\Omega S^{2n+2})^{2g}$. The proof of this last fact arises from an inspection of the pointed mapping space $\mathrm{map}_0(M_g, S^{2n+2})$ which is of the homotopy type of $C(\bar{M}_g; S^{2n})$ [4]. This differential is the only non-trivial one in characteristic zero.

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