On transport twistor spaces

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Outline

Motivation

- Geometric inverse problems in 2 dimensions are often best understood via the following interplay:

$$\text{transport equations} \quad \leftrightarrow \quad \text{fibrewise Fourier analysis}$$

- Transport twistor spaces are complex 2-dimensional manifolds that put these aspects on the same footing.

This talk

- Twistor correspondences – novel point of view for old theorems
- two new theorems that were inspired by twistor considerations

Future

- Twistor spaces as a tool?
- Many intriguing questions about twistor spaces!

Transport equations vs. vertical Fourier analysis

Let (M, g) be an orientable Riemannian surface with (possibly empty) boundary ∂M. Define the unit tangent bundle

$$SM = \{(x, v) \in TM : g(v, v) = 1\}.$$

- **Transport equations.** Let X be the geodesic vector field, $\mathbb{A} \in C^\infty(SM, \mathbb{C}^{n\times n})$ and consider:

$$(X + \mathbb{A})u = f \quad \text{on } SM \tag{TE}$$

Equivalent to a family of ODE:

$$\forall \text{ geodesics } \gamma(t) : \quad \dot{u}(t) + \mathbb{A}(\gamma(t), \dot{\gamma}(t)) \cdot u(t) = 0 \tag{TE'}$$

- **Fibrewise Fourier Analysis.** Any $f \in C^\infty(SM)$ has a unique decomposition into vertical Fourier modes:

$$f = \sum_{k\in\mathbb{Z}} f_k$$

We say that f is **fibrewise holomorphic**, if $f_k = 0$ for $k < 0$.

It is often key to find solutions of the transport equation whose Fourier modes have special properties.

Problem 1: Invariant holomorphic distributions

Find many (distributional) solutions to $Xu = 0$ such that u is fibrewise holomorphic. E.g. one for every chosen lowest Fourier mode!

$\rightsquigarrow$ Tensor tomography problem on closed Anosov surfaces (Paternain–Salo–Uhlmann 2014, Guillarmou 2017)

Problem 2: Matrix holomorphic integrating factors

For which $\mathbb{A}$ does $(X + \mathbb{A})F = 0$ admit a $GL(n, \mathbb{C})$-valued solution F that is fibrewise holomorphic?

$\rightsquigarrow$ Range characterisation for the non-Abelian X-ray transform on simple surfaces (B.-Paternain 2021)

The twistor space of $\mathbb{R}^2$

Let $M = \mathbb{R}^2$, then $SM = \{(z,\mu) \in \mathbb{C}^2 : |\mu| = 1\}$.

Write $z = x + iy$ and $\mu = \cos\theta + i\sin\theta$, then

$$X = \cos\theta \cdot \partial_x + \sin\theta \cdot \partial_y = \mu\partial_z + \bar{\mu}\partial_{\bar{z}} = \bar{\mu}\left(\mu^2\partial_z + \partial_{\bar{z}}\right).$$

Definition

The twistor space of $\mathbb{R}^2$ is $Z = \{(z,\mu) \in \mathbb{C}^2 : |\mu| \leq 1\}$, with (degenerate) complex structure given in terms of the *Cauchy–Riemann* equations

$$(\mu^2\partial_z + \partial_{\bar{z}})f = 0 \quad \text{and} \quad \partial_{\bar{\mu}} f = 0.$$

- Have a 1:1-correspondence:

$$f \in C^\infty(Z) \text{ holomorphic} \leftrightarrow \begin{array}{l}\text{fibrewise holomorphic solution} \\ u \in C^\infty(SM) \text{ to } Xu = 0.\end{array}$$

- Have a holomorphic blow-down map

$$\beta\colon Z \to \mathbb{C}^2, \quad \beta(z,\mu) = (z - \mu^2\bar{z}, \mu),$$

  maps Z° diffeomorphically to a poly-disk in $\mathbb{C}^2$.

Twistor space of an oriented Riemannian surface

The Cauchy–Riemann equations can be encoded in complex vector bundle

$$D = \mathrm{span}_{\mathbb{C}}(\mu^2\partial_z + \partial_{\bar z}, \partial_{\bar\mu}) \subset T_{\mathbb{C}}Z = TZ \otimes \mathbb{C}.$$

This has the following properties:

(i) D is involutive (that is, $[D, D] \subset D$); $[\mu^2\partial_z + \partial_{\bar z}, \partial_{\bar\mu}] = 0$ ✓

(ii) $D \cap \bar D = 0$ on $Z \backslash SM$ and $D \cap \bar D = \mathrm{span}_{\mathbb{C}} X$ on SM; ✓

(iii) the fibres of $Z \to M$ are holomorphic. $\partial_{\bar\mu} \in D$ ✓

Theorem (Existence and uniqueness of twistor space)

Let (M, g) be an oriented Riemannian surface and

$$Z = \{(x, v) \in TM : g(v, v) \le 1\}.$$

Then there exists a unique subbundle $D \subset T_{\mathbb{C}}Z$ of rank 2 with the properties (i),(ii) and (iii). In particular, Z° is a complex surface with $T^{0,1}Z^\circ = D$.

- Quotient $Z/\sim$ is well known (O'Bryan–Rawlsney, LeBrun–Mason, ...), but Z itself seems to have gone unnoticed;
- there are also versions for magnetic flows, etc.

Holomorphic functions on Z

Three algebras of holomorphic functions:

$$\mathcal{A}(Z) \subset \mathcal{A}_{\mathrm{pol}}(Z) \subset \mathcal{A}(Z^\circ)$$

- $\mathcal{A}(Z^\circ) = \{f \in C^\infty(Z^\circ) : f \text{ holomorphic}\}$ (that is, $df|_D = 0$)
- $\mathcal{A}_{\mathrm{pol}}(Z) = \{f \in \mathcal{A}(Z^\circ) : f$ has at most polynomial growth $(\dagger)\}$

$$\exists C, p > 0 : \quad \sup_{(x,v)\in SM} |f(x, rv)| \leq C(1-r)^{-p} \qquad (\dagger)$$

- $\mathcal{A}(Z) = \mathcal{A}(Z^\circ) \cap C^\infty(Z)$

Theorem

$$\mathcal{A}(Z) \cong \{u \in C^\infty(SM) : Xu = 0,\ u \textit{ fibrewise holomorphic}\}$$

- If (M, g) is simple, then $\mathcal{A}(Z)$ is large. By [Pestov–Uhlmann 2005]:

$$\mathcal{A}(Z) \to \mathcal{A}(M), \quad f \mapsto f|_M, \qquad \text{is onto.}$$

- If (M, g) is closed and the geodesic flow is ergodic (e.g. if $K_g < 0$), then

$$\mathcal{A}(Z) \cong \mathbb{C}.$$

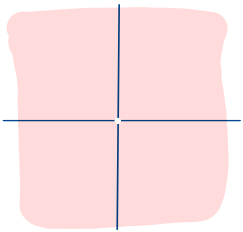
Theorem (B.–Lefeuvre–Paternain)

Let (M,g) be an oriented closed surface. Then

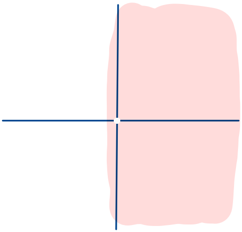
$$\mathcal{A}_{\mathrm{pol}}(Z) \cong \{u \in \mathcal{D}'(SM) : Xu = 0,\ u \textit{ fibrewise holomorphic}\}.$$

In particular, fibrewise holomorphic invariant distributions form an algebra.

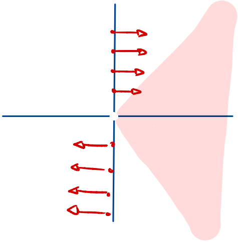
Proof. For u as above, we want to control $\|u_k\|_{C^N}$ as $k \to \infty$. For this we determine $\mathrm{WF}(u)$; in pictures:



Flow invariant: $\mathrm{WF}(u) \subset \mathrm{Char}(X)$



Fibrewise holomorphic: $u = \mathcal{S}u$, $\mathrm{WF}'(\mathcal{S})$ known



Twist property & POS: $\mathrm{WF}(u) \cap \mathbb{H}^* = \emptyset$ □

Moduli space of holomorphic rank n-vector bundles

$$\mathfrak{M}_n(Z) = \left\{ \begin{array}{l} \text{Holomorphic vector bundle structures} \\ \text{on } Z \times \mathbb{C}^n\text{, smooth up to the boundary} \end{array} \right\} / \sim$$

Define

$$\begin{aligned} \mho &= \{\mathbb{A} \in C^\infty(SM, \mathbb{C}^{n\times n}) : \mathbb{A}_k = 0 \text{ for } k < -1)\} \\ \mathbb{G} &= \{F \in C^\infty(SM, GL(n,\mathbb{C})) : F_k = 0 \text{ for } k < 0\}. \end{aligned}$$

Theorem

Let (M, g) be an oriented surface. Then

$$\mathfrak{M}_n(Z) \cong \mho/\mathbb{G},$$

where we quotient by the group action $(\mathbb{A}, F) \mapsto F^{-1}(X + \mathbb{A})F$.

- $\mathfrak{M}_n = \{*\} \;\Leftrightarrow\; \exists$ holomorphic integrating factors for all $\mathbb{A} \in \mho$.

Theorem (TOG principle)

Let (M, g) be a simple surface. Then:

(i) $\mathfrak{M}_1(Z) = \{*\}$ [Salo–Uhlmann 2011]

(ii) $\mathfrak{M}_n(Z) = \{*\}$ *for all* $n \geq 2$ [B.–Paternain]

Proof. Need to show that $\mathbb{G}$ acts transitively on $\mho$:

- Reduce to linear problem with Nash-Moser IFT;
- solve linear problem (+tame estimates) using results on attenuated X-ray transform and microlocal analysis;
- conclude that all orbits are open $\Rightarrow$ action on $\mho$ must be transitive. □

Slogan

The twistor space of a simple surface behaves like a contractible Stein surface.

Future directions

Ongoing work with Monard–Paternain

Produce blow-downs $\beta\colon Z \to \mathbb{C}^2$ for (M, g) *nearly* Euclidean.

Open questions

- Is Z° a Stein surface if (M, g) is simple?
- For which (M, g_1) and (M, g_2) do we have $Z_1 \cong Z_2$?
- Can we deal with the non-ellipticity of CR-equations intrinsically?
- Twistor spaces as a tool?
- …