Subdividing and squeezing

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Abstract

This talk is based on the contents of my Ph.D. thesis. An approach to deciding when a map is 'close' or homotopic to a homeomorphism is to impose restrictions on its point inverses. The idea is that a homeomorphism has point inverses that are precisely points, so if the map has point inverses that are sufficiently 'close' to being points in some suitable sense then the map should be close to a homeomorphism. In this talk contractible point inverses and point inverses that are small in the sense of controlled topology are considered.

Let $f : Y \to X$ be a simplicial map of finite-dimensional locally finite simplicial complexes. The first result is that f has contractible point inverses if and only if it is an ϵ-controlled homotopy equivalence measured in X for all $\epsilon > 0$, if and only if $f \times \mathrm{id} : Y \times \mathbb{R} \to X \times \mathbb{R}$ is a bounded homotopy equivalence measured in the open cone $O(X^+)$ of Pedersen and Weibel.

The whole approach is generalised to geometric algebra, dealing with algebraic images of maps to a fixed finite-dimensional locally finite simplicial complex X. The generalisation uses the X-controlled categories $\mathbb{A}^*(X)$, $\mathbb{A}_*(X)$ of Ranicki and Weiss (1990) and the bounded categories $\mathcal{C}_M(\mathbb{A})$ of Pedersen and Weibel (1989) for M a metric space. Analogous to the barycentric subdivision of a simplicial complex X, an algebraic barycentric subdivision of a chain complex associated to X is defined. The main theorem is that a chain complex C is chain contractible in $\left\{ \begin{array}{c} \mathbb{A}^*(X) \\ \mathbb{A}_*(X) \end{array} \right.$ if and only if "$C \otimes \mathbb{Z}$" $\in \left\{ \begin{array}{c} \mathbb{A}^*(X \times \mathbb{R}) \\ \mathbb{A}_*(X \times \mathbb{R}) \end{array} \right.$ is boundedly chain contractible when measured in $O(X^+)$ for a functor "$- \otimes \mathbb{Z}$" defined appropriately using algebraic subdivision. In the process a squeezing result must be proven: a chain complex with a sufficiently small chain contraction is chain equivalent to one with arbitrarily small chain contractions.

This work has consequences for Poincaré Duality spaces. Squeezing tells us that a PL Poincaré duality space with sufficiently controlled Poincaré duality is necessarily a homology manifold, as homology manifolds are ϵ-controlled Poincaré duality spaces for all ϵ. The main theorem tells us that a PL Poincaré duality space X is a homology manifold if and only if $X \times \mathbb{R}$ has bounded Poincaré duality when measured in the open cone $O(X^+)$. The algebraic subdivision defined should also have an application in passing from the algebraic surgery exact sequence to the controlled.