KLEINE AG: DELIGNE-MUMFORD COMPACTIFICATION

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In this “Kleine AG” we will study the moduli space M_g of stable curves of fixed genus $g \geq 2$, following the work of Deligne and Mumford [DM]. The question we have in mind is that of irreducibility of its geometric fibres. In characteristic 0 the result is classical, and there is a proof, which can be done over $\mathbb{C}$ and is based on analysis of certain coverings of $\mathbb{P}^1(\mathbb{C})$. This method was extended by W. Fulton to positive characteristic using specializations from characteristic 0 to p, but only under the hypothesis $p > 2g + 1$.

We will follow a different approach presented in [DM], considering the compactified moduli stack $\mathfrak{M}_g$ over $\operatorname{Spec} \mathbb{Z}$, allowing also singular degenerations of smooth stable curves, thus linking together the possible different irreducible components. Further, this approach takes into account all characteristics simultaneously and (following ideas of Grothendieck) uses specialization from characteristic 0, where Teichmöller theory can be applied.

There are three important aspects of this approach. The first is that, instead of considering the coarse moduli scheme, Deligne and Mumford work with the fine moduli *stack* of stable curves. Compared with usual schemes, such stacks have the advantage that on the one hand, they give “precise” solutions of moduli problems, and on the other hand they inherit many geometric properties of schemes. We will see the definition and some geometric properties of stacks in the first talk.

The second aspect is the *properness* of the compactified (i. e. with singular “degenerated” curves allowed) moduli stack $\mathfrak{M}_g$. This properness is necessary to reduce to the case of characteristic 0, and is deduced from the stable reduction theorem for curves, which has to be derived from the stable reduction theorem for abelian varieties by relating the stable reduction of a curve with that of its Jacobian. This approach will be presented in the third talk.

The third aspect is that in [DM] one shows even more than the irreducibility of $\mathfrak{M}_g$. More precise, one deals with the moduli stack ${}_G\mathfrak{M}_g$ of stable curves with additional Teichmüller structure of level G (here, G is a finite group), and describes its irreducible components.

All (x.y) without further reference refer to [DM].

The Program

First talk: Deligne-Mumford stacks (45 minutes). The aim of this talk is to give an overview of what a Deligne-Mumford stack is and to present some of its properties, of which we will make use later, without going into technical details. There are two parts: the description of a stack as “an object generalizing schemes and good for solving moduli problems” (plus possibly a more formal definition) and an overview over geometric properties of stacks. The (quite compact) reference is §4 of [DM]. There are clearly other, more detailed references for this talk. For example, a very detailed treatment of stacks can be found in [STACKS], especially

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in the chapters 47, 48. Possibly it will take to much time to give the formal definition, since one would have many trouble (e.g. to define CFG's). However,

- A stack is a category over a given category $\mathfrak{C}$ with a Grothendieck topology,[2] generalizing the notion of a scheme. The fibres of a stack over $\mathfrak{C}$ are groupoids and a category fibred in groupoids over $\mathfrak{C}$ is a stack if and only if isomorphisms form a sheaf and descent data for objects are effective (4.1), the representable stack $\underline{X} = \mathfrak{C}_{/X}$; the Yoneda lemma for stacks; representable morphisms between stacks (4.2). Now the definition of a Deligne-Mumford stack (4.6) can be given.
- In the second part, one should give a feeling of which geometric concepts from the theory of schemes can be carried over to the 2-category of algebraic stacks: what it means for stacks/morphism of stacks to have various (global or of local nature) properties of schemes/morphisms of schemes (p. 100). In particular: separated (4.7), quasi-compact and of finite type; proper (4.11).
- One should explain the notions of connected components of a locally noetherian stack (4.13, 4.14), open and closed substacks, irreducible components of a (noetherian) stack (4.15). As for schemes, the connected components of a normal stack are irreducible (4.16). The number of connected components of a geometric fibre of a proper flat stack over a noetherian scheme is locally constant (4.17). The valuative criterion for separatedness and properness (4.18, 4.19).
- The last feature, which we will need is (4.21), which says that a stack with representable unramified diagonal for which a smooth cover by a representable stack of finite type exists (i. e. an Artin stack with unramified diagonal) is a Deligne-Mumford stack of finite type.

Second talk: Stable curves (45 minutes). The aim of this talk is to define stable curves (not necessarily smooth!), and to establish certain properties of them, which in particular guarantee that the corresponding moduli problem is solved by a Deligne-Mumford stack.

- Give the definition of a stable curve over a scheme S (1.1).
- Let now $\mathfrak{M}_g$ ($g \geq 2$) be the CFG as defined in the beginning of §5. The aim is to prove the theorem (5.1):

  Theorem 0.1. *$\mathfrak{M}_g$ is a separated Deligne-Mumford stack over* $\mathrm{Spec}(\mathbb{Z})$.

  To prove it, follow the steps as described at the beginning of §5:
- First of all, $\mathfrak{M}_g$ is a stack (it is not hard, once one assumes étale descent).
- One needs now a scheme covering $\mathfrak{M}_g$: following (1.2) define the scheme H_g, which represents the "tri-canonically" embedded stable curves of genus g. To obtain it, one has to prove that higher powers of the canonical sheaf $\omega_{C/S}$ on a stable curve C/S are very ample. This is a corollary of the theorem (1.2), which one should prove (possibly one can left out certain technical steps in the proof).
- The next step is to show that the diagonal $\Delta\colon \mathfrak{M}_g \to \mathfrak{M}_g \times \mathfrak{M}_g$ is representable, finite and unramified: this is the theorem (1.11), which one should prove (assuming the technical lemma (1.4), needed for unramifiedness) reformulated in terms of stacks.
- Now the forgetful morphism
  $$H_g \to \mathfrak{M}_g$$
  is representable (since Δ is!), smooth and surjective. All these facts (together with (4.21), needed since $H_g \to \mathfrak{M}_g$ is only smooth, but not étale) imply the above theorem.

[2] one can assume $\mathfrak{C}$ are schemes with étale topology

Third talk: Stable reduction theorem and properties of $\mathfrak{M}_g$ (45 minutes). The aim of this talk is to deduce the stable reduction theorem for curves, and to obtain certain properties of $\mathfrak{M}_g$ and its universal covering $\mathfrak{Z}_g$. In particular, the properness of $\mathfrak{M}_g$ will follow from the stable reduction theorem.

- Let $\mathfrak{Z}_g$ be the "universal curve" and $\mathfrak{M}_g^0$ the smooth locus of $\mathfrak{M}_g$. The result we are aiming is theorem (5.2):

  Theorem 0.2. *The algebraic stacks $\mathfrak{M}_g$ and $\mathfrak{Z}_g$ are proper and smooth over* $\mathrm{Spec}(\mathbb{Z})$ *and the complement of $\mathfrak{M}_g^0$ in $\mathfrak{M}_g$ is a divisor with normal crossings relative to* $\mathrm{Spec}(\mathbb{Z})$.

- The first aspect in the proof is the properness of $\mathfrak{M}_g$, which follows (by an application of the properness criterion in the first talk) from the stable reduction theorem for curves. One should therefore deduce it (2.7) from the analogous result for abelian varieties, quoted in the introduction to [DM].
- Therefore define the two senses of "stable reduction" for curves (2.2) and quote their equivalence (2.3) without proof.[3] Then the direction from Jacobian to curve of theorem (2.4) in the easier case, where the curve has a K-rational point,[4] follows from the result of Raynaud (2.5).
- The second aspect concerns the property of $\mathfrak{M}_g - \mathfrak{M}_g^0$ to be a divisor with normal crossings. This follows from the analogous property for the smooth covering scheme H_g of $\mathfrak{M}_g$, and is proven in §1 by using deformation theory to analyze the local analytic structure of H_g. Probably, it will be too much to do this analysis, so one can just quote the result.

Fourth talk: Teichmüller structures (45 minutes). The aim of this talk is to introduce the Teichmüller structure on a stable curve, and to prove that the number of irreducible components of a geometric fibre of the stack ${}_G\mathfrak{M}_g$ of stable curves with Teichmüller structure of fixed level, is constant. More concrete:

- Let G be a finite group. A Teichmüller structure on a stable *smooth* curve X/S of level G is essentially just a surjective exterior homomorphism from the fundamental group $\pi_1(X/S)$ to G (5.5-5.6).
- However, one has to be careful with the order of G and the characteristic of the base scheme S: the stack ${}_G\mathfrak{M}_g^0$ classifying curves with a Teichmüller structure of level G is only defined over $\mathrm{Spec}\,\mathbb{Z}[\frac{1}{n}]$, where $n = \mathrm{ord}(G)$.
- The forgetful morphism ${}_G\mathfrak{M}_g^0 \to \mathfrak{M}_g^0[\frac{1}{n}]$ is representable, finite and étale (5.7-5.8). In fact, Teichmüller structures on X/S define a sheaf on $(\mathrm{Sch}/S)_{et}$, which by results of Grothendieck is representable by an étale covering of S.
- Up to now we were restricted to smooth curves. We resolve this problem by considering the normalization ${}_G\mathfrak{M}_g$ of $\mathfrak{M}_g[\frac{1}{n}]$ with respect to ${}_G\mathfrak{M}_g^0$, which is still defined only over $\mathrm{Spec}\,\mathbb{Z}[\frac{1}{n}]$. Now we see the important theorem (5.9):

  Theorem 0.3. *The geometric fibres of ${}_G\mathfrak{M}_g \to$* $\mathrm{Spec}\,\mathbb{Z}[\frac{1}{n}]$ *are normal, and ${}_G\mathfrak{M}_g^0$ is fibrewise dense in ${}_G\mathfrak{M}_g$.*

- Its proof makes use of the local[5] description of ${}_G\mathfrak{M}_g - {}_G\mathfrak{M}_g^0$ and of ${}_G\mathfrak{M}_g$ provided by the Abhyankar-Artin lemma (which holds in a much more general situation). A necessary

[3] this is reasonable, since we only need one direction, from "sense 1" to "sense 2", which proof is technical enough, to just left it out.

[4] this is the only case required to apply the properness criterion

[5] for the étale topology

condition to apply it is given by the third talk, from which follows that ${}_G\mathfrak{M}_g - {}_G\mathfrak{M}_g^0$ is a divisor with normal crossings.

- Once the theorem is established, it follows (using the result on semicontinuity (4.17) from the first talk) from the properness and flatness of ${}_G\mathfrak{M}_g^0 \to \operatorname{Spec} \mathbb{Z}[\frac{1}{n}]$, that its geometric fibres (being normal!) have all the same number of connected components. This is the reason, why in the next talk it is enough to compute the number of connected components of ${}_G\mathfrak{M}_g^0 \times \operatorname{Spec} \mathbb{C}$.

Fifth talk. Connected components of ${}_G\mathfrak{M}_g^0$ (45 minutes). In this talk, the main theorem, which describes the set of connected components of a geometric fibre of ${}_G\mathfrak{M}_g^0$, will be deduced. Its proof makes use of the Teichmüller theory over $\mathbb{C}$.

- By the results of the last talk every geometric fibre of the projection ${}_G\mathfrak{M}_g^0 \to \operatorname{Spec} \mathbb{Z}[\frac{1}{n}]$ has the same number of connected (or, equivalently, irreducible) components. Hence to count this number, it is enough to consider ${}_G\mathfrak{M}_g^0 \times \operatorname{Spec} \mathbb{C}$. To the coarse moduli scheme underlying this moduli stack, the classical results from the theory of Teichmüller spaces can be applied:
- Let Π denote the fundamental group of a fixed Riemannian surface of genus g (5.12). Then a Teichmüller curve of genus g is just a Riemannian surface C of genus g over $\mathbb{C}$ plus an exterior (that means modulo the action of Π on itself by conjugation) isomorphism of its (topological) fundamental group $\pi_1(C)$ with Π.
- Now, the Teichmüller theory, which ideas should briefly be explained (as, for example, presented in [We]) , says that the space T_g, classifying Teichmüller curves of genus g is homeomorphic to a ball, hence connected. From this one obtains the theorem (5.13):

  Theorem 0.4. *The number of connected components of any geometric fibre of the projection of ${}_G\mathfrak{M}_g^0$ onto $\operatorname{Spec} \mathbb{Z}[\frac{1}{n}]$ is equal to the number of orbits of $\operatorname{Aut}^0(\Pi)$ in the set of the exterior epimorphisms from Π to G.*

- Now, the irreducibility of the geometric fibres of $\mathfrak{M}_g$ is just the special case $G = 1$ of the above theorem.

References

[DM] Deligne P., Mumford D.: The irreducibility of the space of curves of given genus

[STACKS] The stacks project authors. Stacks project. `http://math.columbia.edu/algebraic_geometry/stacks-git`

[We] Weil A.: Modules des surfaces de Riemann, Sém. Bourbaki, 168, 1957-58.