ARithmetische Geometrie OberSeminar

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Topic: Shtukas and the Taylor expansion of L-functions

The aim of the seminar is to work through the preprint of Z. Yun and W. Zhang [1]. In this paper, the authors define *Heegner–Drinfeld cycles* on the moduli stack of *Drinfeld Shtukas of rank two with r-modifications* for an even integer r. They prove an identity between

(1) the r-th central derivative of the quadratic base change L-function associated to an everywhere unramified cuspidal automorphic representation π of PGL_2 and
(2) the self-intersection number of the π-isotypic component of the Heegner–Drinfeld cycle.

This identity can be viewed as a function-field analog of the Waldspurger and the Gross–Zagier formulae for higher derivatives of L-functions.

The proof relies on the relative trace formula associated to the diagonal torus in PGL_2. It is best encapsulated in the following diagram [1, (1.8)] which compares various distributions on the unramified Hecke algebra for PGL_2. The vertical lines mean equality up to the factor $(\log q)^r$.

$$\begin{tikzcd}
\Sigma_{u\in\mathbb{P}^1(F)-\{1\}}\ \mathbb{J}_r(u,f) \arrow[rr, equal, "2"] \arrow[d, dash, "6+7"] & & \mathbb{J}_r(f) \arrow[rr, equal, "8"] \arrow[d, dash] & & \Sigma_\pi\ \mathbb{J}_r(\pi,f) \arrow[d, dash, "9"] \\
\Sigma_{u\in\mathbb{P}^1(F)-\{1\}}\ \mathbb{I}_r(u,f) \arrow[rr, equal, "3+4+5"] & & \mathbb{I}_r(f) \arrow[rr, equal, "9"] & & \Sigma_{\mathfrak{m}}\ \mathbb{I}_r(\mathfrak{m},f)
\end{tikzcd}$$

Here, $\mathbb{J}_r$ is some analytically defined distribution on the Hecke algebra. The summands $\mathbb{J}_r(\pi,\cdot)$ on the right are related to the r-th central derivative of the (quadratic base change) L-function associated to the unramified cuspidal representation π.

The distribution $\mathbb{I}_r$ is defined via intersection theory on the moduli stack of PGL_2-shtukas with r legs. Namely

$$\mathbb{I}_r(f) := \big(\theta_*[\mathrm{Sht}_T^r], f * \theta_*[\mathrm{Sht}_T^r]\big)$$

where $\theta_*[\mathrm{Sht}_T^r]$ is the Heegner-Drinfeld cycle on this stack and where f acts through its Hecke correspondence.

The aim is now to relate the π-isotypic components of these distributions on the right hand side of the diagram. This is done by comparing the summands on the left hand side for sufficiently many test functions. The numbers indicate the talk corresponding to the respective comparison.

Talks

0. Talk: Introduction

Explanation of main results.

1. Talk: The moduli stack of G-Shtukas

Follow §5.1–5.3: Define the Hecke stack and the moduli stack of Shtukas. Then define G-Shtukas for $G = \mathrm{PGL}_2$. Define Hecke correspondences (esp. [1, Remark 5.11]). Use without comment the properties of ${}_c\mathrm{CH}_{2r}(\mathrm{Sht}^r_G \times \mathrm{Sht}^r_G)$.

2. Talk: The distributions $\mathbb{I}_r(f)$ and $\mathbb{J}_r(f)$ on the Hecke algebra

This talk has two commuting parts. Follow §5.4–5.5 to define T-shtukas and the Heegner-Drinfeld cycle $\theta_*[\mathrm{Sht_T}] \in {}_\mathrm{c}\mathrm{Ch_r}(\mathrm{Sht^r_G})_\mathbb{Q}$. Then define the distribution $\mathbb{I}_r(f)$ as in Def. 5.15.

Present the content of §2. In particular, introduce the distribution $\mathbb{J}(f,s)$ with derivatives $\mathbb{J}_r(f)$. You can omit the proofs in §2.3–2.5. Finally explain the orbital integral decomposition [1, (2.17)]. Note that one aim of the seminar is to prove Thm. 9.2.

3. Talk: Intersection theory

Follow Appendix A: Explain the concept of a cohomological correspondence (A.4). We especially need Prop. A.12 (related to Varshavsky's proof of the Deligne conjecture), and Prop. A.10.

4. Talk: Orbital decomposition of the intersection number

Follow §6.2 to prove Theorem 6.5, using Proposition A.12, but modulo Theorem 6.6.

5. Talk: Extension of the fundamental cycle

Follow §6.3: Complete the proof of the previous talk by proving Theorem 6.6, using Proposition A.10. Possibly omit the proof of the dimension statements in Lemma 6.10.

6. Talk: Geometric interpretation of the orbital integrals $\mathbb{J}(\gamma, f, s)$

Follow §3 and prove Proposition 3.2 and Corollary 3.3.

7. Talk: The key identity $\mathbb{I}_r(f) = (\log q)^{-r}\mathbb{J}_r(f)$

Follow §8, in particular explain the proof of Proposition 8.1.

8. Talk: Analytic spectral decomposition

Follow §4: Introduce $\mathbb{J}_\pi(f,s)$. Prove Proposition 4.5 and Theorem 4.7, accepting on faith Theorem 4.3.

9. Talk: Cohomological spectral decomposition and end of proof

Present Theorem 7.14 and Corollary 7.15. Then finish the proof of Theorems 1.6 and 1.2, see §9. You will have to make some choices regarding the details you present. (Yun and Zhang suggest to take the results of §7.2 up to Definition 7.12 on faith.)

References

[1] Z. Yun, W. Zhang, *Shtukas and the Taylor expansion of L-functions*, arXiv:1512.02683